Hodge Index Theorem by Linear Algebra

1 Linear algebra

Proposition 1. Let V be a real vector space of dimension n with a non-degenerated symmetric bilinear form $\langle -,- \rangle$. Consider the subset

$$S = \{x \in V \mid \langle x, x \rangle > 0\}.$$

Suppose that $S \neq \varnothing$ and there exists $z \in V$ such that $z^\perp \cap S = \varnothing$, then the signature of $\langle -,- \rangle$ is of type $(1, n-1)$, where $z^\perp = \{v \in V \mid \langle z, v \rangle = 0\}$.

Proof. Choose $h \in S$. We claim that the restriction of $\langle -,- \rangle$ on $h^\perp$ is negative definite. First, the restriction is non-degenerated. Otherwise, note that h and $h^\perp$ generate V. If there exists $0 \neq x \in h^\perp$ such that $\langle x, y \rangle = 0$ for all $y \in h^\perp$, then in particular $\langle x, h \rangle = 0$, thus $x \in h^\perp$ and $x \in V^\perp = \{0\}$, which is a contradiction.

Hence if the restriction is not negative definite, then there exists $x \in h^\perp$ such that $\langle x, x \rangle > 0$. Then consider the subspace V_0 generated by h and x. We have $h^2, x^2 > 0$ and $h \cdot x = 0$, thus the restriction of $\langle -,- \rangle$ on V_0 is of type $(2, 0)$. Hence $z^\perp \cap V_0 = \{0\}$. However, consider the dimension count

$$\dim(z^\perp + V_0) = \dim(z^\perp) + \dim(V_0) - \dim(z^\perp \cap V_0) = (n-1) + 2 - 0 = n + 1 > n = \dim(V),$$

which is a contradiction. □

Remark 2. Geometrically, we have the following equivalent statement:

(a) $S \neq \varnothing$ and there exists $z \in V$ such that $z^\perp \cap S = \varnothing$;

(b) the signature of $\langle -,- \rangle$ is of type $(1, n-1)$;

(c) the set S has two connected components.

We have shown $(a) \Rightarrow (b)$ in Proposition 1. If the signature of $\langle -,- \rangle$ is of type $(1, n-1)$, then we can choose a basis such that the matrix of $\langle -,- \rangle$ is

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}.$$

Then the set S is given by the equation

$$x_1^2 - x_2^2 - \cdots - x_n^2 > 0,$$

which has two connected components, thus $(b) \Rightarrow (c)$. Finally, if S has two connected components, then for any $z \in S$, we claim that $z^\perp \cap S = \varnothing$. Otherwise, there exists $x \in z^\perp \cap S$. Considering on

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the subspace V_0 generated by z and x, we see that z and $-z$ lie in the same connected component of S. For every $y \in S$, assume $z.y > 0$ (otherwise replace z by $-z$), then the line segment $tz + (1-t)y$ for $t \in [0,1]$ connects z and y in S. Hence S is path connected, which is a contradiction.

Example 3. Let $V = \mathbb{R}^4 = \{(t,x,y,z) \mid t,x,y,z \in \mathbb{R}\}$ be the Minkowski space with the bilinear form

$$\langle (t,x,y,z),(t',x',y',z')\rangle = tt' - xx' - yy' - zz'.$$

Then (the closure of) the set $S = \{(t,x,y,z) \in V \mid t^2 - x^2 - y^2 - z^2 > 0\}$ is called the set of *light cone*. It has two connected components, which are called the *future light cone* and the *past light cone*; see fig. 1.

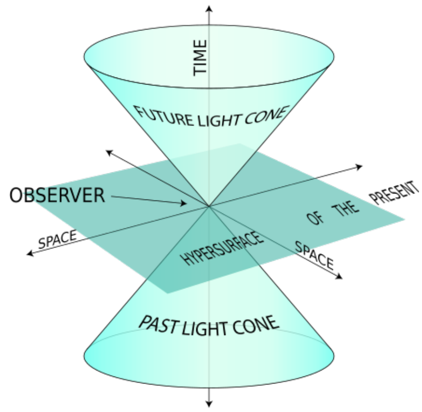

Figure 1: Light cone in Minkowski space, from https://en.wikipedia.org/wiki/Light_cone

2 Hodge index theorem for surfaces

Lemma 4 (Riemann-Roch theorem for surfaces)**.** Let X be a smooth projective surface over an algebraically closed field $\Bbbk$ and D a divisor on X. Then we have

$$h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) + h^0(\mathcal{O}_X(K_X - D)) = \chi(\mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X),$$

where K_X is the canonical divisor of X.

Lemma 5. Let X be a smooth projective surface over an algebraically closed field $\Bbbk$ and D a divisor on X. If $D^2 > 0$, then at least one of D and $-D$ is pseudo-effective.

Proof. Suppose for contradiction that both D and $-D$ are not pseudo-effective. In particular, we have $h^0(\mathcal{O}_X(mD)) = 0$ for all $m > 0$. By Lemma 4, we have

$$h^0(\mathcal{O}_X(K_X - mD)) \geq \chi(\mathcal{O}_X) + \frac{1}{2} mD \cdot (mD + K_X) > 0 \text{ for all } m \gg 0.$$

Hence there exist effective divisors $E_m \sim K_X - mD$ for all $m \gg 0$. We have $-D \sim_{\mathbb{Q}} \frac{1}{m}(E_m - K_X)$ is pseudo-effective, which is a contradiction. □

Theorem 6. Let X be a smooth projective surface over an algebraically closed field $\Bbbk$. Then the intersection form on $\mathrm{NS}(X)_{\mathbb{R}} = \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is of type $(1, \rho(X) - 1)$, where $\rho(X) = \dim_{\mathbb{R}} \mathrm{NS}(X)_{\mathbb{R}}$ is the Picard number of X.

Proof. Note that both $\mathrm{Psef}(X) \setminus \{0\}$ and $-\mathrm{Psef}(X) \setminus \{0\}$ are convex cones in $\mathrm{NS}(X)_{\mathbb{R}}$ and they are disjoint. Hence there exists a hyperplane H in $\mathrm{NS}(X)_{\mathbb{R}}$ such that $H \cap \mathrm{Psef}(X) = \{0\}$ by the geometric form of Hahn-Banach theorem. By Lemma 5,

$$H \cap \{D \in \mathrm{NS}(X)_{\mathbb{R}} \mid D^2 > 0\} = \varnothing.$$

Then the conclusion follows from Proposition 1. □

3 Siu's inequality in the surface case

Theorem 7 (Siu's inequality)**.** Let X be a smooth projective variety of dimension n over an algebraically closed field $\Bbbk$. Let A, B be nef divisors on X such that $A^n > 0$. Then we have

$$A^n \cdot B^n \leq n(A^{n-1} \cdot B) \cdot (B^{n-1} \cdot A).$$

In the surface case, it is easy. The following is a proof using Theorem 6 and linear algebra.

Proposition 8. Let V be a real vector space of dimension n with a non-degenerated symmetric bilinear form $\langle -, - \rangle$ of type $(1, n-1)$. Let $v \in V$ with $\langle v, v \rangle > 0$. Then for any $w \in V$, we have

$$\langle v, v \rangle \cdot \langle w, w \rangle \leq \langle v, w \rangle^2,$$

and the equality holds if and only if v and w are linearly dependent.

Proof. By normalizing v, we may assume $\langle v, v \rangle = 1$. Consider the decomposition $V = \mathbb{R}v \oplus v^{\perp}$. For any $w \in V$, we can write $w = av + u$ for some $a \in \mathbb{R}$ and $u \in v^{\perp}$. It is equivalent to show that

$$a^2 + 2a\langle v, u \rangle + \langle u, u \rangle \leq (a + \langle v, u \rangle)^2,$$

which is equivalent to $\langle u, u \rangle \leq \langle v, u \rangle^2 = 0$. Note that the restriction of $\langle -, - \rangle$ on $v^{\perp}$ is negative definite. The conclusion follows. □

Remark 9. Proposition 8 is a question in the postgraduate entrance exam of East China Normal University in 2025.