Moduli of dynamics on projective spaces

The references for this section are [Sil12; Lev11]. For some basic knowledge of moduli problems and geometric invariant theory, I refer to [Fan+06; HM98; MFK94].

Throughout this section, let $\mathbf{k}$ be an arbitrary field of characteristic zero and $\Bbbk$ be its algebraic closure unless otherwise specified.

1 Review of algebraic geometry by functors

In this subsection, we review some facts about the viewpoint of functors in algebraic geometry.

First, recall that we can associate any scheme X with a contravariant functor $\mathbf{h}_X$ to sets. Let us state it in a general setting.

Definition 1. Let $\mathbf{S}$ be a category. For any $X \in \mathrm{Obj}(\mathbf{S})$, we can define a functor $\mathbf{h}_X : \mathbf{S}^{\mathrm{op}} \to \mathbf{Set}$ by

$$\begin{aligned} &\mathbf{h}_X(T) = \mathrm{Arr}(T, X), && \forall T \in \mathrm{Obj}(\mathbf{S}), \\ &\mathbf{h}_X(a) : \mathrm{Arr}(T, X) \to \mathrm{Arr}(S, X),\ b \mapsto b \circ a, && \forall (a : S \to T) \in \mathrm{Arr}(S, T). \end{aligned}$$

For any $f : X \to Y$, we can associate it with a natural transformation $\mathbf{h}_f : \mathbf{h}_X \to \mathbf{h}_Y$ by

$$\mathbf{h}_f(T) : \mathbf{h}_X(T) \to \mathbf{h}_Y(T), \quad b \mapsto f \circ b.$$

The amazing Yoneda lemma tells us that this functor $X \mapsto \mathbf{h}_X$ does not lose any information.

Theorem 2 (Yoneda Lemma, see [Sta26, Tag 001P]). Let $\mathbf{S}$ be a category. The functor

$$\mathbf{h}__ : \mathbf{S} \to \mathrm{Fun}(\mathbf{S}^{\mathrm{op}}, \mathbf{Set}), \qquad X \mapsto \mathbf{h}_X, \quad f \mapsto \mathbf{h}_f$$

is fully faithful, that is, the map

$$\mathrm{Arr}_{\mathbf{S}}(X, Y) \to \mathrm{Nat}(\mathbf{h}_X, \mathbf{h}_Y)$$

is a bijection of sets for any $X, Y \in \mathrm{Obj}(\mathbf{S})$.

Definition 3. Let $\mathbf{S}$ be a category. A functor $F : \mathbf{S}^{\mathrm{op}} \to \mathbf{Set}$ is said to be representable if it lies in the full subcategory $\mathbf{h}__(\mathbf{S})$ up to isomorphism.

Recall that, tautologically, a functor $F : \mathbf{S}^{\mathrm{op}} \to \mathbf{Set}$ is a presheaf F over $\mathbf{S}$ valued in $\mathbf{Set}$. For some reason, we will use the terminology $\mathbf{PSh}_{\mathbf{S}}(\mathbf{Set})$ or simply $\mathbf{PSh}_{\mathbf{S}}$.

Let us focus on the case of schemes. For a scheme X, we will often identify X with the functor $\mathbf{h}_X$. In other words, we will write $X(T) = \mathrm{Mor}(T, X)$. In particular, if X is of finite type over a field $\mathbf{k}$ and $\mathbf{K}/\mathbf{k}$ is a field extension, then $X(\mathbf{K}) = \mathrm{Mor}_{\mathbf{k}}(\mathbf{K}, X)$ is the set of $\mathbf{K}$-points in X. This coincides with the usual notation. It is worth noting that if X is a scheme over a base scheme S and T/S, then $X(T) = \mathrm{Mor}_S(T, X) = \mathrm{Mor}_T(T, X_T) = X_T(T)$ under the identification induced by $X_T \to X$ using the universal property of fiber products. In particular, if $S = \mathrm{Spec}\,\mathbf{k}$ and $T = \mathrm{Spec}\,\Bbbk$, we have $X(\Bbbk) = X_{\Bbbk}(\Bbbk)$. And from this viewpoint, an

element in $X(T)$ should be viewed as a section of $X_T \to T$.

Remark 4. One should view the Yoneda embedding $\mathbf{h}_{_} : \mathbf{S} \to \mathbf{PSh}_{\mathbf{S}}$ as the embedding $\mathscr{C}([0,1]) \hookrightarrow L^2([0,1])$. Many objects do not exist as schemes, but we can first define them as functors and then try to figure out when they are indeed schemes. This is like when we try to solve a PDE: we cannot find a smooth solution at the beginning, but we can first find an L^2-weak solution and then try to show that it is smooth.

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$$\mathbf{h}_{_} : \mathbf{S} \to \mathbf{PSh}_{\mathbf{S}} \quad \longleftrightarrow \quad \mathscr{C}([0,1]) \hookrightarrow L^2([0,1]).$$

We also need the notions of group objects and group actions in the language of category theory. From now on, we always assume the category $\mathbf{S}$ has fiber products and a terminal object. And by an easy computation, one can see that $\mathbf{h}_{_}$ preserves fiber products and terminal objects.

Definition 5 (Group objects). Let $\mathbf{S}$ be a category with fiber products and a terminal object. A *group object* consists of an object $G \in \mathrm{Obj}(\mathbf{S})$ and a factorization of functors

$$\mathbf{h}_G : \mathbf{S}^{\mathrm{op}} \to \mathbf{Grp} \xrightarrow{\text{forget}} \mathbf{Set}.$$

Remark 6. Group objects can also be described by arrows as in [MFK94, Definition 0.1].

Let $* \in \mathrm{Obj}(\mathbf{S})$ be the terminal object in $\mathbf{S}$. Given a factorization as in Definition 5, $\mathbf{h}_G(*)$ is a group. In particular, it has an identity element $e : * \to G$. And we have natural transformations $h_{G\times G} = h_G \times h_G \to h_G$ and $h_G \to h_G$ (with target **Set**) by the multiplication and inverse laws of groups. By the Yoneda lemma, they give arrows $m : G \times G \to G$ and $i : G \to G$.

Conversely, if one has $e : * \to G$, $m : G \times G \to G$ and $i : G \to G$ satisfying the commutative diagrams in [MFK94, Definition 0.1], then applying $\mathrm{Arr}(T, -)$ gives the group law on the set $G(T)$. This gives the factorization required in Definition 5.

Definition 7 (Group actions and quotients). Let $\mathbf{S}$ be a category with fiber products and a terminal object, G be a group object in $\mathbf{S}$ and $X \in \mathrm{Obj}(\mathbf{S})$. A *group action* of G on X is an arrow $G \times X \to X$ such that for every T, $G(T) \times X(T) \to X(T)$ is a usual group action on sets.

Let G be a group object with actions $\sigma : G \times X \to X$ and $\tau : G \times Y \to Y$. A *G-equivariant arrow* is an arrow $a : X \to Y$ with the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\mathrm{id}\times a} & G \times Y \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} \\ X & \xrightarrow{a} & Y \end{array}$$

commutative.

Consider an action $\sigma : G \times X \to X$ of G on X. The *categorical quotient* of X by G is an object Y together with a G-invariant morphism $\pi : X \to Y$ such that for any other object Z (with trivial G-action) and any G-invariant morphism $f : X \to Z$, there exists a unique morphism $\tilde{f} : Y \to Z$ making the following diagram commute:

$$\begin{array}{ccccc} G \curvearrowright X & \xrightarrow{\pi} & Y & \curvearrowleft & \mathrm{id} \\ & \searrow^{f} & \downarrow{\scriptstyle \tilde{f}} & & \\ & & Z & \curvearrowleft & \mathrm{id} \end{array}$$

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Example 8. Let $\mathbf{S}$ be a category and G be a group object in $\mathbf{PSh}_{\mathbf{S}}$. In $\mathbf{PSh}_{\mathbf{S}}$, the categorical quotient always exists. Indeed, it is very easy to describe since the target is $\mathbf{Set}$. Let $\sigma : G \curvearrowright F$ be a group action in $\mathbf{PSh}_{\mathbf{S}}$. Then the quotient is given by $(F/G)(T) \coloneqq F(T)/G(T)$ on an object T. And for an arrow $a : S \to T$ in $\mathbf{S}$, we have the commutative diagram

$$\begin{array}{ccc} G(T) \times F(T) & \xrightarrow{\sigma(T)} & F(T) \\ {\scriptstyle G(a)\times F(a)}\downarrow & & \downarrow{\scriptstyle F(a)} \\ G(S) \times F(S) & \xrightarrow[\sigma(S)]{} & F(S). \end{array}$$

Hence for $F(a) : F(T) \to F(S)$, any $G(T)$-orbit maps into a $G(S)$-orbit. Then we get a well-defined map of sets $F(T)/G(T) \to F(S)/G(S)$. Define this map to be $(F/G)(a)$. The universal property is easy to check. Hence we get the quotient functor. However, in general, the properties of this quotient are very bad and we will not use it.

Definition 9. Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$ via $\sigma : G \times X \to X$ or $\sigma^{\#} : A \to \Gamma(A \otimes \mathcal{O}_G, X \times G) \cong A \otimes \Gamma(\mathcal{O}_G, G)$. The *invariant ring* is defined as

$$A^G \coloneqq \left\{ f \in A \,\middle|\, \sigma^{\#}(f) = f \otimes 1 \in A \otimes \Gamma(\mathcal{O}_G, G) \right\}.$$

Remark 10. Suppose that $G(\mathbf{k})$ is dense in G. Then the above invariant ring A^G coincides with the usual set-theoretic invariant ring $A^{G(\mathbf{k})}$. For simplicity, here we assume $G = \operatorname{Spec} H$ is linear (in the general case, note that the action of G will factor through its affinization).

First we always have $A^G \subset A^{G(\mathbf{k})}$ since the action of $g \in G(\mathbf{k})$ on A is given by

$$g^{\#} : A \xrightarrow{\sigma^{\#}} A \otimes H \xrightarrow{g^{\#}: H \to \mathbf{k}} A \otimes \mathbf{k} = A.$$

On the other hand, if $f \in A^{G(\mathbf{k})}$ with $G(\mathbf{k})$ dense, then $\sigma^{\#}(f)$ coincides with $f \otimes 1$ on a dense subset and hence they are the same.

Example 11. Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. Then $Y = \operatorname{Spec} A^G$ is the categorical quotient in the category of affine schemes. If G is reductive, it is also the categorical quotient in the category of varieties; see [MFK94, Proof of Theorem 1.1].

Remark 12. In Example 11, if G is reductive, then Y is of finite type over $\mathbf{k}$; see Theorem 18. If G is a general algebraic group, then A^G may fail to be of finite type over $\mathbf{k}$. For some examples, see [Fre01].

In algebraic geometry, we have another notion to describe the quotient with a more geometric flavor.

Definition 13 (Geometric quotients for schemes). Let G be an algebraic group acting on a variety X via $\sigma : G \times X \to X$. Given a categorical quotient $\pi : X \to Y$, we say that Y is a *geometric quotient* of X by G if the following conditions hold:

(a) The variety Y is equipped with the quotient topology induced by π, i.e., a subset $U \subseteq Y$ is open if and only if $\pi^{-1}(U)$ is open in X.

(b) The morphism π is surjective, and its geometric fibers are single G-orbits as sets of geometric points: for every algebraically closed extension $\Omega/\mathbf{k}$ and $y \in Y(\Omega)$, we have $\pi^{-1}(y)(\Omega) = G(\Omega) \cdot x$ for some $x \in X(\Omega)$.

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(c) The sheaf of regular functions on Y is isomorphic to the sheaf of G-invariant regular functions on X, i.e., for any open subset $U \subseteq Y$, we have $\mathcal{O}_Y(U) \cong \mathcal{O}_X(\pi^{-1}(U))^G$.

Note that here we require that a geometric quotient is a categorical quotient. In [MFK94], this is not necessary.

Example 14. Let G be a finite group. Then there is a discrete algebraic group $G = \sqcup_{g \in G} \operatorname{Spec} \mathbf{k}$ with $G(\mathbf{k}) = G$. Let X be a projective variety over $\mathbf{k}$ with a G-action. Then the categorical quotient X/G exists and is a geometric quotient; see [Liu02, Exercise 3.3.23].

In general, a categorical quotient may not be a geometric quotient, and the existence of quotients is not guaranteed.

Example 15. Let $G = \mathbb{G}_m$ act on $X = \mathbb{A}^1$ by scaling, i.e., $t \cdot x = tx$ for $t \in \mathbb{G}_m$ and $x \in \mathbb{A}^1$. The categorical quotient of this action is a single point, since any G-invariant morphism from $\mathbb{A}^1$ to another variety must be constant. We can easily see that the categorical quotient is not a geometric quotient, since the fibers of the quotient map are not precisely the G-orbits in X.

The most interesting case is when G is a reductive group. For some basic knowledge of reductive groups, see Remark 16 and Example 17.

Remark 16. Recall that any affine algebraic group G is a linear algebraic group. For a linear algebraic group G and $g \in G$, we say that g is *semisimple* if it is diagonalizable over the algebraic closure of the base field, and g is *unipotent* if all its eigenvalues are equal to 1. We have the *Jordan decomposition* of g as $g = g_s g_u$, where g_s is semisimple and g_u is unipotent, and they commute with each other.

On the other hand, G has a unique maximal normal solvable connected subgroup, which is called the *radical* of G, denoted by $R(G)$. All unipotent elements of $R(G)$ form a normal subgroup of G, called the *unipotent radical* of G, denoted by $R_u(G)$. The unipotent radical $R_u(G)$ can also be characterized as the unique maximal normal unipotent connected subgroup of G. We say that G is *reductive* if its unipotent radical is trivial, i.e., $R_u(G) = \{e\}$, where e is the identity element of G. This is equivalent to saying that the radical $R(G)$ is a torus, i.e., $R(G)_{\Bbbk} \cong (\mathbb{G}_{m,\Bbbk})^n$ for some $n \geq 0$.

Example 17. The general linear group GL_n is reductive, since its radical is the center of GL_n, which is isomorphic to $\mathbb{G}_m$.

To see this, we may first base change to $\Bbbk$. Let $R = R(\mathrm{GL}_n)$ be the radical of GL_n. By Lie-Kolchin theorem and solvability, R is conjugate to a subgroup of the group B of upper triangular matrices in GL_n. Since R is normal in GL_n, it is contained in the intersection of all conjugates of B, which is the group of scalar matrices, i.e., $R \subseteq Z(\mathrm{GL}_n) \cong \mathbb{G}_m$. Note that we always have $Z(G)^\circ \subseteq R(G)$ for any linear algebraic group G, where $Z(G)^\circ$ is the connected component of the identity in the center of G. Hence R is a torus, and thus GL_n is reductive.

By a similar argument, one can show that $R(\mathrm{PGL}_n) = \{e\}$, and thus PGL_n is also reductive.

We have the following theorem to show the power of reductive groups and for later use.

Theorem 18 (ref. [MFK94, Theorem 1.1 and Amplification 1.3]). Let G be a reductive group acting on an affine variety $X = \operatorname{Spec} A$ over $\mathbf{k}$. Then the categorical quotient $X \to X/G$ exists and $X/G = \operatorname{Spec} A^G$ is also an affine variety (in particular, of finite type) over $\mathbf{k}$. And it is a geometric quotient if and only if all $G_{\Bbbk}$-orbits of geometric points in $X_{\Bbbk}$ are closed.

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Example 19. Let $M_n = \mathbb{A}^{n^2}$ be the variety of matrices of size n and let $G(\mathbf{K}) = \mathrm{GL}_n(\mathbf{K})$ act on $M_n(\mathbf{K})$ by conjugation for any field extension $\mathbf{K}/\mathbf{k}$. The variety M_n is affine and we denote it by $M_n = \operatorname{Spec} A$ with $A = \mathbf{k}[a_{11}, \cdots, a_{nn}]$. Then we have $A_{\mathbf{K}}^{G(\mathbf{K})} = \mathbf{K}[\sigma_1, \cdots, \sigma_n]$, where $\sigma_k = \operatorname{Tr}(\Phi^k)$ and $\Phi = (a_{ij})_{ij}$ is the matrix with indeterminate entries and Φ^k is the k-th matrix power. Set $E_n = \operatorname{Spec} B \cong \mathbb{A}_{\mathbf{k}}^n$ with $B = \mathbf{k}[\sigma_1, \cdots, \sigma_n]$. The above equality describes its base change to $\mathbf{K}$. Then $E_n = M_n/\operatorname{GL}_n$ is the categorical quotient.

This quotient is not a geometric quotient since it only captures the eigenvalues rather than the Jordan form.

Remark 20. For the projective case, there is also a whole theory to deal with quotients by reductive groups, namely, the *geometric invariant theory*. The standard reference is [MFK94].

We also need *Luna's étale slice theorem*.

Theorem 21 (Luna's étale slice theorem, see [Lun73; BKS25]). Work over $\Bbbk$. Let G be a reductive group acting on an affine variety X via σ with quotient $\pi : X \to Y$. Suppose that $x \in X$ has stabilizer G_x and a closed orbit. Then there exists a locally closed affine G_x-stable subset $S \subset X$ containing x such that

(a) the morphism $\varphi : G \times^{G_x} S \to X, (g, s) \mapsto \sigma(g)(s)$ is G-equivariant with open and affine image, where $G \times^{G_x} S := (G \times S)/G_x$; the G_x action on $G \times S$ is given by right translation on G and σ on S; the G-action on $G \times^{G_x} S$ is given by left translation;

(b) the diagram

$$\begin{array}{ccc} G \times^{G_x} S & \xrightarrow{\varphi} & X \\ {\scriptstyle /G}\downarrow & & \downarrow{\scriptstyle \pi} \\ S/G_x & \xrightarrow{\bar{\varphi}} & Y \end{array}$$

is a fiber product and both φ and $\bar{\varphi}$ are étale.

2 Review of moduli problem and Hilbert schemes

A moduli space is a geometric space whose points represent isomorphism classes of certain geometric objects. For example, fix an algebraically closed field $\Bbbk$; all elliptic curves over $\Bbbk$ can be classified by the j-invariant, which gives a bijection between isomorphism classes of elliptic curves and elements of $\mathbb{A}^1(\Bbbk)$. However, this classification is just a set-theoretic one. We would like to have a geometric "parameter space" such that we can "deform" the objects continuously. This is the initial motivation for moduli spaces.

In algebraic geometry, "deforming objects continuously" is usually described by flat families. Hence the most perfect object to represent this moduli problem is such a flat family $\mathcal{X} \to M$, where M is a variety parameterizing all elliptic curves, and the fiber over each point $m \in M(\Bbbk)$ is the elliptic curve corresponding to m. Furthermore, we hope that this family is universal, i.e., any other flat family $\mathcal{Y} \to T$ of elliptic curves is obtained by pulling back $\mathcal{X} \to M$ along a unique morphism $T \to M$. (In this example, however, such a perfect family does not exist; we will discuss this later.) It can be described in the language of functors. Consider the functor

$$\mathcal{M} : \mathbf{sch}_{\Bbbk}^{\mathrm{op}} \to \mathbf{Set}, \quad T \mapsto \{\text{flat families of elliptic curves over } T\}/\sim,$$

Here a family of elliptic curves means a smooth proper morphism of relative dimension one with geometrically connected genus one fibers and a section. The equivalence relation $\sim$ is given by isomorphisms over T preserving the sections. If the family $\mathcal{X} \to M$ above exists, then it represents the functor $\mathcal{M}$. In this case,

we say that the moduli problem $\mathcal{M}$ is *representable*, M is called a *fine moduli space* and $\mathcal{X} \to M$ is called a *universal object*.

Hence, to study a moduli problem, we follow these steps:

(a) Define the moduli functor $\mathcal{M}$.

(b) Check whether $\mathcal{M}$ is representable.

Slogan

Moduli problem of parameterizing objects in Σ	$\longleftrightarrow$	*Find a functor* $\mathcal{M}$ *with* $\mathcal{M}(\Bbbk) = \Sigma$ *describing "deformations";*
Algebraic families over T *with fibers in* Σ	$\longleftrightarrow$	*elements in* $\mathcal{M}(T)$*;*
Pullback / base change along $f : T' \to T$	$\longleftrightarrow$	*the map* $\mathcal{M}(f) : \mathcal{M}(T) \to \mathcal{M}(T')$*.*

Here we recall the most important example: Hilbert schemes, which parameterize the subschemes of a fixed scheme X with a fixed Hilbert polynomial.

Theorem 22 (Hilbert schemes, see [Fan+06, Theorem 5.14]). Let X be a projective variety over $\mathbf{k}$, $\mathcal{H}$ an ample line bundle and P a polynomial with $\mathbb{Q}$-coefficients. The *Hilbert functor* is defined by

$$\mathcal{H}i\ell\mathcal{b}_{(X,\mathcal{H},P)/\mathbf{k}}(T) := \left\{ \mathcal{V} \subset X \times T \;\middle|\; \pi : \mathcal{V} \to T \text{ is flat, } \chi(\mathcal{O}_{\mathcal{V}|_t} \otimes \mathcal{H}^{\otimes m}) = P(m) \text{ for all } t \in T \right\}.$$

It is representable and represented by a scheme projective over $\mathbf{k}$, called the *Hilbert scheme*, denoted by $\mathrm{Hilb}_{(X,\mathcal{H},P)/\mathbf{k}}$.

We list many applications of Hilbert schemes.

Example 23 (Hilbert scheme of points). Let X be a projective variety over $\mathbf{k}$ and $\mathcal{H}$ an ample line bundle. Consider n distinct $\mathbf{k}$-points $x_1, \cdots, x_n$ in X as a reduced subscheme $Z \subset X$. Then its Hilbert polynomial is $\chi(\mathcal{O}_Z \otimes \mathcal{H}^{\otimes m}) = h^0(\mathcal{O}_Z \otimes \mathcal{H}^{\otimes m}) = n$. More generally, every zero-dimensional subscheme of length n has the same Hilbert polynomial, including nonreduced ones. Hence they form a Hilbert scheme $\mathrm{Hilb}^n X := \mathrm{Hilb}_{(X,\mathcal{H},n)/\mathbf{k}}$, called the *Hilbert scheme of n points on X*. It represents the functor of finite flat families $\mathcal{Z} \subset X \times T$ of degree n over T.

There is also a related geometric construction. Let $Y = X \times \cdots \times X$ be the product of n copies of X with an action of the discrete group $\mathfrak{S}_n$ by permutation. By Example 14, the quotient $Y/\mathfrak{S}_n$ exists. Denote it by $\mathrm{Sym}^n X$. Its geometric points parameterize effective zero-cycles of degree n on X. There is a canonical *Hilbert–Chow morphism*

$$\mathrm{Hilb}^n X \to \mathrm{Sym}^n X, \quad Z \mapsto \sum_{x \in \mathrm{Supp}(Z)} \mathrm{length}(\mathcal{O}_{Z,x})[x],$$

where the formula is taken over $\Bbbk$; see [Fan+06, Theorem 7.1.14].

In general, this morphism is not an isomorphism since the cycle does not record the scheme structure of Z. For example, on an affine chart $\mathbb{A}^2 \subset \mathbb{P}^2$ with coordinates u, v, the two subschemes $V(v, u^2)$ and $V(u, v^2)$ are distinct but both map to $2[(0,0)]$. However, if X is a smooth projective curve, then the Hilbert–Chow morphism is an isomorphism and $\mathrm{Hilb}^n X \cong \mathrm{Sym}^n X$.

Example 24 (Dual projective spaces). Let $H = \mathbb{P}(V) \cong \mathbb{P}^n$ for some vector space V. Recall that H parameterizes the hyperplanes in V. And dually, the space $L = \mathbb{P}(V^\vee)$, which parameterizes lines in V, is also isomorphic to $\mathbb{P}^n$. This "duality" can be described by a Hilbert scheme: all hyperplanes in H form a

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Hilbert scheme L with universal family $\mathcal{U} \to L$ given by

$$\mathcal{U} = \{(h, l) \in H \times L \mid h \in \mathcal{U}_l \subset H \times \{l\}\}.$$

Later we will need a generalization as follows. Let $X \to S$ be a projective smooth morphism with geometric fibers $\mathbb{P}^n$ and noetherian base S. Set $\mathcal{H} \cong \omega_{X/S}^{\otimes -1}$, a relatively ample line bundle. With respect to $\mathcal{H}$, we can define the Hilbert scheme $X^\vee \to S$ parameterizing the hyperplanes on $X \to S$ with a universal family $\mathcal{U}_S$ which gives the above $\mathcal{U}$ over each fiber.

Example 25 (Schemes of morphisms). Let X, Y be projective varieties over $\mathbf{k}$, $\mathcal{H}$ an ample divisor on $W = X \times Y$ and P a polynomial with Q-coefficients. Set

$$\mathcal{M}or_{(X,Y;\mathcal{H},P)/\mathbf{k}}(T) := \left\{ f : X \times T \to Y \times T \;\middle|\; \chi_W(\mathcal{O}_{\Gamma_{f_t}} \otimes \mathcal{H}^{\otimes m}) = P(m) \text{ for all } t \in T \right\}.$$

The functor $\mathcal{M}or_{(X,Y;\mathcal{H},P)/\mathbf{k}}$ is represented by an open subscheme of $\mathrm{Hilb}_{(W,\mathcal{H},P)/\mathbf{k}}$; see [Fan+06, Theorem 5.23].

Here we give a brief description. Set $H = \mathrm{Hilb}_{(W,\mathcal{H},P)/\mathbf{k}}$ with universal family $\mathcal{U} \subset W \times H = X \times Y \times H$ and $M = \{t \in H \mid \mathcal{U}_t \to X \times \{t\} \text{ is an isomorphism }\}$. We only need to show that M is an open subset of H. If so, M has a natural open subscheme structure and a universal family $X \times M \cong \mathcal{U}|_M \to Y \times M$ and the natural factorization

$$\begin{array}{ccc} \mathcal{M}or_{(X,Y;\mathcal{H},P)/\mathbf{k}} & \xhookrightarrow{f \mapsto \Gamma_f} & \mathcal{H}ilb_{(W,\mathcal{H},P)/\mathbf{k}} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \cong} \\ M(T) & \hookrightarrow & H(T) \end{array}$$

shows that ϕ is a natural isomorphism (the diagram gives ϕ and shows injectivity, while the universal family shows surjectivity) and hence $\mathcal{M}or_{(X,Y;\mathcal{H},P)/\mathbf{k}}$ is represented by M.

Suppose that M is nonempty and $t \in M$.

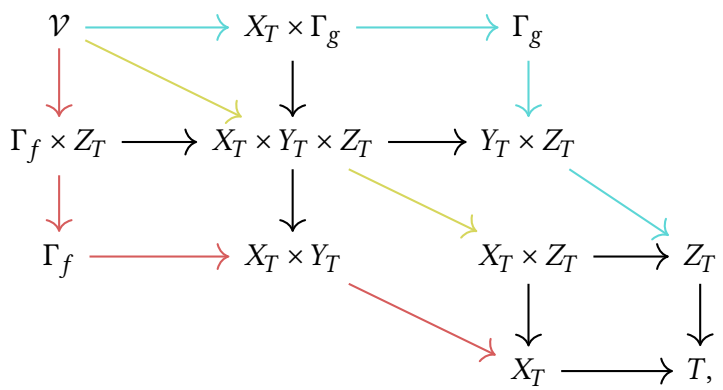
Denote by $p : \mathcal{U} \to X_H$ the projection to X over H. By the local criterion of flatness (see [Fan+06, Lemma 5.21(3)]), p is flat at every point of $\mathcal{U}_t$. The nonflat locus is closed in $\mathcal{U}$. Since $\mathcal{U} \to H$ is proper, its image is closed and avoids t. Hence, after replacing H by an open neighborhood V of t, $p_V : \mathcal{U}_V \to X_V$ is flat. Similarly, the locus where p is not quasi-finite is closed in $\mathcal{U}$, and its proper image in H avoids t. After shrinking V again, p_V is proper and quasi-finite, hence finite flat. The rank of $(p_V)_* \mathcal{O}_{\mathcal{U}_V}$ is locally constant on X_V and equals one on X_t. The locus where the rank differs from one is closed in X_V. Removing its image under the proper morphism $X_V \to V$, we may assume that the rank is one everywhere. Then the unit map $\mathcal{O}_{X_V} \to (p_V)_* \mathcal{O}_{\mathcal{U}_V}$ is an isomorphism, so p_V is an isomorphism. This shows that M is open, and the same argument gives $\mathcal{U}_M \cong X_M$. Let Z be another projective variety over $\mathbf{k}$, $\mathcal{H}'$, $\mathcal{H}''$ be ample line bundles on $Y \times Z$ and $X \times Z$ respectively and P' another polynomial. Then the composition

$$\mathcal{M}or_{(X,Y;\mathcal{H},P)/\mathbf{k}} \times \mathcal{M}or_{(Y,Z;\mathcal{H}',P')/\mathbf{k}} \to \mathcal{M}or_{(X,Z;\mathcal{H}'',P_1,\cdots,P_n)/\mathbf{k}}, \quad ([f],[g]) \mapsto [g \circ f]$$

is a morphism. Here $\mathcal{M}or_{(X,Z;\mathcal{H}'',P_1,\cdots,P_n)/\mathbf{k}}$ means the disjoint union of $\mathcal{M}or_{(X,Z;\mathcal{H}'',P_i)/\mathbf{k}}$ for $i = 1, \cdots, n$.

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For any $T/\mathbf{k}$ and $f \in \mathcal{M}or_{(X,Y;\mathcal{H},P)/\mathbf{k}}(T)$, $g \in \mathcal{M}or_{(Y,Z;\mathcal{H}',P')/\mathbf{k}}(T)$, consider the diagram over T



where $\mathcal{V}$ is given by the scheme-theoretic intersection of $X_T \times \Gamma_g$ and $\Gamma_f \times Z_T$, i.e., their fiber product. Note that $X_T \times_T \Gamma_g \to X_T \times_T Y_T$ is an isomorphism since $\Gamma_g \to Y_T$ is. It follows that $\mathcal{V} \to \Gamma_f$ is an isomorphism. And $\Gamma_f \to X_T$ is also an isomorphism. Then $\mathcal{V} \to X_T$ is an isomorphism. Then by the universal property of graphs, we see that the yellow arrow $\mathcal{V} \to X_T \times Z_T$ is isomorphic to the graph $\Gamma_{g\circ f} \to X_T \times Z_T$. And $\mathcal{V} \to T$ is flat since $X_T \to T$ is. Suppose that its Hilbert polynomials with respect to $\mathcal{H}''$ are $\{P_1, \cdots, P_n\}$. (It may have multiple polynomials since T may have multiple connected components.) Then by the universal property of $\mathcal{M}or_{(X,Z;\mathcal{H}'',P_1,\cdots,P_n)/\mathbf{k}}$, the family $\mathcal{V}$ induces a morphism $T \to \mathrm{Mor}_{(X,Z;\mathcal{H}'',P_1,\cdots,P_n)/\mathbf{k}}$ of schemes. Let $T = \mathrm{Mor}_{(X,Y;\mathcal{H},P)/\mathbf{k}} \times \mathrm{Mor}_{(Y,Z;\mathcal{H}',P')/\mathbf{k}}$ and let f, g be the pullbacks of the universal families along the canonical projections; we get the desired morphism

$$\mathrm{Mor}_{(X,Y;\mathcal{H},P)/\mathbf{k}} \times \mathrm{Mor}_{(Y,Z;\mathcal{H}',P')/\mathbf{k}} \to \mathrm{Mor}_{(X,Z;\mathcal{H}'',P_1,\cdots,P_n)/\mathbf{k}} \quad ([f],[g]) \mapsto [g \circ f].$$

Example 26 (Schemes of endomorphisms). Let $X = Y$ and $\mathcal{H}_W = \mathcal{H} \boxtimes \mathcal{H}$ on $W = X \times X$ with $\mathcal{H}$ ample on X in Example 25. For an endomorphism $f : X \to X$, the Hilbert polynomial of $i : X \cong \Gamma_f \subset W$ with respect to $\mathcal{H}_W$ is given by

$$\begin{aligned}\chi_W(\mathcal{O}_{\Gamma_f} \otimes \mathcal{H}_W^{\otimes m}) &= \chi_W(i_*\mathcal{O}_X \otimes p_1^*\mathcal{H}^{\otimes m} \otimes p_2^*\mathcal{H}^{\otimes m}) = \chi_W\left(i_*\left(\mathcal{O}_X \otimes \mathcal{H}^{\otimes m} \otimes f^*\mathcal{H}^{\otimes m}\right)\right) \\ &= \chi_X(\mathcal{H}^{\otimes m} \otimes f^*\mathcal{H}^{\otimes m}).\end{aligned}$$

Suppose further that X is smooth (to use GRR) and $\mathbf{k} = \Bbbk$ is algebraically closed (only to ensure $\mathrm{NS}(X)$ is stable as a set under base field change). By Grothendieck-Riemann-Roch, the latter is totally determined by the intersection numbers of $\mathcal{H}$, $f^*\mathcal{H}$ and the Todd class of X. In particular, if one fixes the Néron-Severi class $[f^*\mathcal{H}] = [\mathcal{L}] \in \mathrm{NS}(X)$, then the Hilbert polynomial of Γ_f is uniquely determined. On the other hand, if f, g belong to a connected flat family, that is, there is $(\Phi : X_T \to Y_T) \in \mathcal{M}or_{(X,Y,*)/\mathbf{k}}(T)$ with T connected and $t, s \in T(\mathbf{k})$ such that $\Phi_t \cong f$, $\Phi_s \cong g$, then $f^*\mathcal{H} \cong \Phi_t^*\mathcal{H}$ is algebraically equivalent to $g^*\mathcal{H} \cong \Phi_s^*\mathcal{H}$. That is, $[f^*\mathcal{H}] = [g^*\mathcal{H}] \in \mathrm{NS}(X)$. Hence,

$$\mathcal{E}nd_{(X,\mathcal{H},[\mathcal{L}])/\mathbf{k}}(T) \coloneqq \{f : X \times T \to X \times T \mid f_t \curvearrowright X \times \{t\},\ [f_t^*\mathcal{H}] = [\mathcal{L}] \in \mathrm{NS}(X \times \{t\})\}$$

defines a representable functor $\mathcal{E}nd_{(X,\mathcal{H},[\mathcal{L}])/\mathbf{k}}$, and its scheme consists of some connected components of $\mathrm{Mor}_{(X,X,\mathcal{H}',P)/\mathbf{k}}$ for some $\mathcal{H}'$ and P.

In the case $X = \mathbb{P}^n$, we have a canonical choice of $\mathcal{H}$ and $[\mathcal{L}]$ is uniquely determined by the degree d of the endomorphisms. Then we get a representable functor $\mathcal{E}nd_{\mathbb{P}^n,d}$. By directly writing down the coordinates, we can describe its scheme as

$\mathrm{End}_{\mathbb{P}^n,d} \cong \mathbb{P}^N \setminus V(R), N = (n+1)\binom{n+d}{d} - 1, R$ is the resultant of polynomials defining endomorphisms .

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An important fact is that R is given by a homogeneous polynomial; see [Sil12, Theorem 1.8] for some references. We will use this fact later.

Example 27 (Automorphism group fixing an ample class). Let X be a projective variety over $\mathbf{k}$ and $\mathcal{H}$ an ample line bundle. Its *moduli functor of automorphisms with polarization* $\mathcal{H}$ is defined as $\mathcal{A}ut_{(X,\mathcal{H})/\mathbf{k}}(T) := \mathcal{E}nd_{(X,\mathcal{H},[\mathcal{H}])/\mathbf{k}}(T)$.

It is represented by $Aut_{(X,\mathcal{H})/\mathbf{k}}$ with universal family $f : X \times Aut_{(X,\mathcal{H})/\mathbf{k}} \to X \times Aut_{(X,\mathcal{H})/\mathbf{k}}$. This morphism f is indeed an isomorphism. To see this, consider a geometric fiber $f_t : X_t \to X_t$. Since $f_t^*\mathcal{H}$ is ample, f_t is finite and surjective. Since $f_t^*\mathcal{H}$ is algebraically equivalent to $\mathcal{H}$, their Hilbert polynomials agree. For sufficiently large m, both $\mathcal{H}^{\otimes m}$ and $f_t^*\mathcal{H}^{\otimes m}$ are very ample and have no higher cohomology. Hence pullback gives an isomorphism $H^0(X_t, \mathcal{H}^{\otimes m}) \cong H^0(X_t, f_t^*\mathcal{H}^{\otimes m})$. The corresponding projective embeddings show that f_t is an isomorphism. Then the fiberwise isomorphism criterion used in Example 25 shows that f is an isomorphism. Then, combining Example 25, we see that $\mathrm{Aut}_{(X,\mathcal{H})/\mathbf{k}}$ is an algebraic group. And the universal family $f : X \times Aut_{(X,\mathcal{H})/\mathbf{k}} \to X \times Aut_{(X,\mathcal{H})/\mathbf{k}} \to X$ gives a canonical action of $\mathrm{Aut}_{(X,\mathcal{H})/\mathbf{k}}$ on X.

In the case $X = \mathbb{P}^n$, we do not need to choose $\mathcal{H}$ and $\mathrm{Aut}_{\mathbb{P}^n} \cong \mathrm{PGL}_{n+1}$ by Example 26.

The main moduli functor we will care about is the moduli functor of dynamics on $\mathbb{P}^n$. From now on, for dynamics we assume $n \geq 1$ and $d \geq 2$.

Definition 28. The functor of d-polarized dynamics on $\mathbb{P}^n$ is defined as

$$\mathcal{D}_{d\curvearrowright n}(T) := \left\{(f : X \to X)/T \;\middle|\; X \to T \text{ smooth projective with geometric fiber } \mathbb{P}^n,\ f_t^*\mathcal{O}(1) \cong \mathcal{O}(d)\right\} / \sim,$$

where $(f : X \to X) \sim (f' : X' \to X')$ if there exists an isomorphism $g : X \to X'$ over T such that $f' \circ g = g \circ f$. This is a functor on $\mathbf{sch}_{\mathbb{Z}}^{\mathrm{op}}$, and we will freely change the base to $\mathbf{k}$ with the same notation.

However, as in many moduli problems we care about, the fine moduli space may not exist. An important situation is that if there is some "isotrivial" but not trivial family, then it cannot be obtained by pulling back the universal family. And hence, $\mathcal{M}$ is not representable. For an example of isotrivial but not trivial family, see Remark 37.

Hence, there is a weaker version, called *coarse moduli space*, satisfying the following conditions:

- there is a $1:1$ correspondence between the isomorphism classes parameterized by $\mathcal{M}$ and the closed points in M;
- for every "algebraic family $F \to T$ whose fibers are objects parameterized by $\mathcal{M}$", there exists a unique morphism $f : T \to M$ induced by F such that F_t is isomorphic to the object parameterized by $f(t) \in M$.

In the rigorous language of functors, "an algebraic family over T" means an element F in $\mathcal{M}(T)$; a morphism corresponds to an element $f_F \in \mathrm{Mor}(T, M) = h_M(T)$. Hence a coarse moduli space should consist of a scheme M and a natural transformation $\eta : \mathcal{M} \to h_M$ with $\eta(T) : \mathcal{M}(T) \to h_M(T), F \mapsto f_F$. The first condition above is just $\eta(\Bbbk) : \mathcal{M}(\Bbbk) \to h_M(\Bbbk)$ being bijective.

The remaining issue is the uniqueness of M. For example, if we have a morphism $g : M \to N$ which is bijective on geometric points, then $(N, \mathbf{h}_g \circ \eta)$ also gives the same pointwise classification. Here $\mathbf{h}_g : h_M \to h_N$ is the natural transformation given by composition with g. Thus the pointwise classification does not determine the scheme structure. Hence we require the coarse moduli space (M, η) to have the universal property that every natural transformation $\xi : \mathcal{M} \to \mathbf{h}_N$ factors uniquely through η.

In summary, we have the following definition.

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Definition 29. Let $\mathcal{M}$ be a moduli functor over $\mathbf{k}$. A *coarse moduli space* of $\mathcal{M}$ is a pair (M, η) with M a $\mathbf{k}$-scheme of finite type and $\eta : \mathcal{M} \to \mathbf{h}_M$ a natural transformation such that

(a) for any other pair $(N, \xi : \mathcal{M} \to \mathbf{h}_N)$, there exists a unique factorization

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\eta} & \mathbf{h}_M \\ & \searrow^{\xi} & \downarrow \exists! \\ & & \mathbf{h}_N. \end{array}$$

(b) for every algebraically closed extension $\Omega/\mathbf{k}$, $\eta(\Omega) : \mathcal{M}(\Omega) \to \mathbf{h}_M(\Omega) = M(\Omega)$ is a bijection;

Here we only consider $\mathbf{k}$-schemes (essentially) of finite type.

By the universal property, if the coarse moduli space exists, then it is unique.

Remark 30. Let $\mathcal{M}$ be a moduli functor. Suppose that $\mathcal{M}$ has a fine moduli space, that is, $\mathcal{M}$ is representable and represented by a scheme M of finite type. Then we easily see that M is also a coarse moduli space of $\mathcal{M}$.

Conversely, suppose that $\mathcal{M}$ has a coarse moduli space (M, η). Then M is a fine moduli space iff η is a natural isomorphism. This is also equivalent to the condition that $\eta(T)$ is injective for all T and there exists an element $\mathcal{X} \in \mathcal{M}(M)$ which gives the identity morphism $\eta(M)(\mathcal{X}) = \mathrm{id}_M \in h_M(M)$.

Example 31. Let us see an example of a moduli functor with the universal property in Definition 29 but with no coarse moduli space. Consider

$$\mathcal{M}(T) = \{(f : \mathcal{E} \to \mathcal{E})/T \mid \mathcal{E} \text{ vector bundle of rank } n \text{ over } T\} / \sim,$$

where $(f : \mathcal{E} \to \mathcal{E}) \sim (f' : \mathcal{E}' \to \mathcal{E}')$ if there exists an isomorphism $g : \mathcal{E} \to \mathcal{E}'$ over T such that $f' \circ g = g \circ f$. Also denote by $M_n \cong \mathbb{A}^{n^2}$ the variety of matrices of size n.

Set $E_n = M_n / \mathrm{GL}_n \cong \mathbb{A}^n$ as in Example 19 and $\eta : \mathcal{M} \to \mathbf{h}_{E_n}$ by

$$\eta(T) : \mathcal{M}(T) \to \mathbf{h}_{E_n}(T), \quad (\mathcal{E}, f) \mapsto \mathrm{Eigen}(f) = (\mathrm{Tr}\, f, \mathrm{Tr}\, f^2, \cdots, \mathrm{Tr}\, f^n).$$

Since f locally is given by $f_U = (f_{U,ij})_{ij} : \mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus n}$ with $f_{U,ij} \in \mathcal{O}_T(U)$, $\mathrm{Tr}\, f$ is indeed a well-defined morphism.

Given any pair $(N, \xi : \mathcal{M} \to \mathbf{h}_N)$, let us construct the factorization $\tilde{\xi} : \mathbf{h}_{E_n} \to \mathbf{h}_N$. First the universal family $\mathcal{U}$ on M_n, more explicitly, $\mathcal{U} = (a_{ij})_{ij} : \mathcal{O}_{M_n}^{\oplus n} \to \mathcal{O}_{M_n}^{\oplus n}$ with $\mathcal{O}_{M_n} = \mathbf{k}[a_{11}, \cdots, a_{nn}]$, gives a morphism $M_n \to \mathcal{M}$. Composing with $\mathcal{M} \to E_n$, we get the quotient map in Example 19. Then we have

$$\begin{array}{ccccc} M_n & \xrightarrow{\mathcal{U}} & \mathcal{M} & \xrightarrow{\eta} & E_n \\ & & & \searrow^{\xi} & \downarrow \tilde{\xi} \\ & & & & N \end{array}$$

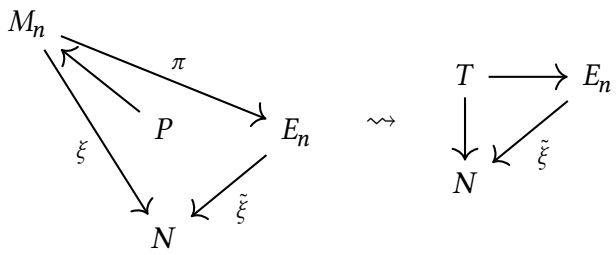
We need to check the commutativity of the right triangle.

Fix $T/\mathbf{k}$ and $(f : \mathcal{E} \to \mathcal{E}/T) \in \mathcal{M}(T)$. Let $p : P \to T$ be the scheme over T parameterizing isomorphisms between $\mathcal{O}_T^{\oplus n}$ and $\mathcal{E}$. It is a GL_n-torsor since locally $\mathcal{O}_U^{\oplus n} \cong \mathcal{E}_U$ and we have a universal family $\mathcal{I} : \mathcal{E}_P \cong \mathcal{O}_P^{\oplus n}$. Then we get

$$f' : \mathcal{O}_P^{\oplus n} \xrightarrow{\mathcal{I}^{-1}} \mathcal{E}_P \xrightarrow{f} \mathcal{E}_P \xrightarrow{\mathcal{I}} \mathcal{O}_P^{\oplus n}.$$

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Then it induces a morphism $P \to M_n$. This morphism is GL_n-equivariant, where the action on the left is given by translation on P and on the right is given by conjugation. We get the diagram on the left



and taking the quotient by GL_n, we get the diagram on the right , which is commutative.

However, E_n is not a coarse moduli space since $\mathcal{M}(\Bbbk)$ is the set of Jordan forms of matrices of size n and $E_n(\Bbbk)$ only captures the eigenvalues.

Remark 32. The above construction in Example 31 has a very general formulation, namely *quotient stacks*. But we will not explore this theory. I only want to emphasize that in the above construction, the fact that the object (vector bundle) is locally trivial is very important. Hence in the general theory, we need to equip the base category $\mathbf{S}$ with some Grothendieck topology.

3 Automorphism groups and coarse moduli

First we show that the automorphism group of a dynamical system $f : \mathbb{P}^n \to \mathbb{P}^n$ is finite.

Definition 33. Let $f \in \mathrm{End}_{(\mathbb{P}^n,d)/\mathbf{k}}(\mathbf{K})$. Then its automorphism group is defined as an algebraic subgroup of PGL_{n+1} over $\mathbf{K}$ by

$$\mathrm{Aut}_f(T) := \{g \in \mathrm{PGL}_{n+1}(T) \mid g \circ f_T = f_T \circ g\}.$$

Since the condition $g \circ f_T = f_T \circ g$ is algebraic and $\mathrm{End}_{(\mathbb{P}^n,d)/\mathbf{k}}$ is separated, the functor Aut_f is representable by an algebraic subgroup of PGL_{n+1}.

Proposition 34. Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a surjective endomorphism with dynamical degree $d \geq 2$ over a field $\mathbf{k}$. Then its automorphism group is finite, that is,

$$\#\,\mathrm{Aut}_f(\Bbbk) < \infty.$$

Proof. Here we take a shortcut. WLOG, we may base change to $\Bbbk$ and then assume the base field is algebraically closed. By [Fak03; BD01; DS10], the periodic points of f are Zariski dense in $\mathbb{P}^n$. Hence we can take $x_1, \cdots, x_{n+1} \in \mathrm{Per}(f)$ whose lifts to $\Bbbk^{n+1}$ are linearly independent and $x_{n+2} \in \mathrm{Per}(f)$ whose lift has a nonzero coordinate for all $i = 1, \cdots, n+1$ with respect to the basis $x_1, \cdots, x_{n+1}$. If an automorphism $g \in \mathrm{Aut}_{\mathbb{P}^n}(\Bbbk) = \mathrm{PGL}_{n+1}(\Bbbk)$ fixes all $x_1, \cdots, x_{n+2}$, then g must be the identity on $\mathbb{P}^n$.

There exists $N \geq 1$ such that $x_i \in \mathrm{Per}_N(f)$ for all $i = 1, \cdots, n+2$. Note that $\mathrm{Per}_N(f)$ is a finite set. And $\forall x \in \mathrm{Per}_N(f)$,

$$f^N(g(x)) = g(f^N(x)) = g(x).$$

Hence $g(\mathrm{Per}_N(f)) \subset \mathrm{Per}_N(f)$ and then $g(\mathrm{Per}_N(f)) = \mathrm{Per}_N(f)$. Hence we find a homomorphism

$$\mathrm{Aut}_{f/\mathbf{k}}(\Bbbk) \hookrightarrow \mathrm{Permutation}(\mathrm{Per}_N(f)).$$

It is injective since $x_i \in \mathrm{Per}_N(f)$. The latter is finite and hence so is $\mathrm{Aut}_{f/\mathbf{k}}(\Bbbk)$. □

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Example 35. Consider the power map $f : z \mapsto z^d$ for $d \geq 3$. Then $\mu_{d-1} \subset \mathrm{Aut}_{f/\Bbbk}(\Bbbk)$ since $m_\xi \circ f \circ m_{\xi^{-1}}(z) = \xi \cdot \xi^{-d} z^d = z^d = f(z)$ for $\xi \in \mu_{d-1}$ and m_a is the map given by multiplication by a.

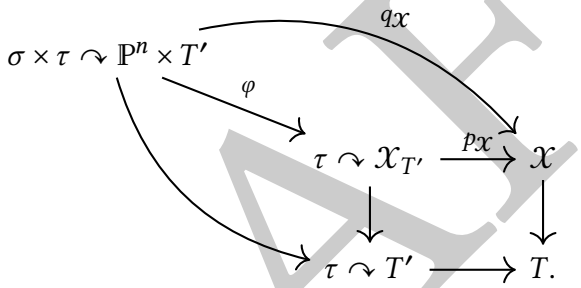
Example 36. More generally, let $f : [x_0 : \cdots : x_n] \mapsto [x_0^d : \cdots : x_n^d]$. Then any involution $\tau \in \mathfrak{S}_{n+1}$, i.e., a permutation of $n+1$ symbols satisfying $\tau^2 = \mathrm{id}$, induces an automorphism $g_\tau \in \mathrm{Aut}_{\mathbb{P}^n}(\mathbf{k})$ by $[x_0 : \cdots : x_n] \mapsto [x_{\tau(0)} : \cdots : x_{\tau(n)}]$ which commutes with f. In particular, f has a non-trivial automorphism group and $\mathcal{D}_{d\curvearrowright n}$ can't have a fine moduli space by Remark 37.

Remark 37. Let $\mathcal{M}$ be a moduli functor. If some element $X \in \mathcal{M}(\Bbbk)$ has a "non-trivial automorphism group", it is usually easy to construct an "isotrivial but not trivial family". Let us illustrate it using $\mathcal{D}_{d\curvearrowright n}$ as an example.

Let $f \in \mathcal{D}_{d\curvearrowright n}(\Bbbk)$ with $\sigma \in \mathrm{Aut}_f(\Bbbk)$ of order two. Let T' be an elliptic curve with a nonzero 2-torsion point τ and $q_T : T' \to T = T'/\tau$. Consider $G = \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{P}^n \times T'$ by $\sigma \times \tau$ and denote the quotient by $q_{\mathcal{X}} : \mathbb{P}^n \times T' \to \mathcal{X}$. The morphisms $f \times \mathrm{id} \curvearrowright \mathbb{P}^n \times T'$ and $\mathbb{P}^n \times T' \to T'$ are equivariant and descend to

$$f : \mathcal{X} \to \mathcal{X}, \quad \mathcal{X} \to T.$$

Pulling back the family $\mathcal{X} \to T$ along $G \curvearrowright T' \to T$, we get a diagram



Since both $q_{\mathcal{X}}$ and $p_{\mathcal{X}}$ are étale covers of degree 2, φ is an isomorphism. Then $\mathcal{X} \to T$ is isotrivial. And we easily see that the family f is also isotrivial. Suppose that $\mathcal{X} \cong T \times \mathbb{P}^n$ is trivial. Then we get a diagram

$$\begin{array}{ccc} \mathbb{P}^n \times T' & \xrightarrow{\sigma\times\tau} & \mathbb{P}^n \times T' \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ \mathbb{P}^n \times T' & \xrightarrow{\mathrm{id}\times\tau} & \mathbb{P}^n \times T'. \end{array}$$

Note that φ is over T', so $\varphi \in \mathrm{Aut}_{\mathbb{P}^n}(T') = \mathrm{PGL}_{n+1}(T') = \mathrm{PGL}_{n+1}(\Bbbk)$ since T' is proper. Then $\varphi = \phi \times \mathrm{id}$ and the commutative diagram gives $(\phi \circ \sigma) \times \tau = \phi \times \tau$. That is, $\phi \circ \sigma = \phi$ and then $\sigma = \mathrm{id}$. This contradicts the condition. Hence $\mathcal{X}$ is not trivial and neither is f.

Viewing T' as a G-torsor over T, we have $\mathcal{X} = \big(T' \times_T (\mathbb{P}^n \times T)\big)/G =: T' \times^G (\mathbb{P}^n \times T)$, where the nontrivial element of G acts by translation by τ on T' and by σ on $\mathbb{P}^n$. This construction is classical and is called the *associated bundle*.

Now let us turn to constructing the coarse moduli space of dynamics on $\mathbb{P}^n$ by GIT.

Theorem 38. The coarse moduli space $D_{d\curvearrowright n}$ of the moduli functor $\mathcal{D}_{d\curvearrowright n}$ exists and is an affine variety.

Proof.

Step 1. First we set $D = D_{d\curvearrowright n} = \mathrm{End}_{\mathbb{P}^n,d} / \mathrm{PGL}_{n+1}$ and show that this exists and is a geometric quotient.

By Example 26, $\mathrm{End}_{\mathbb{P}^n,d} = \mathbb{P}^N \setminus H$ with H an ample divisor. Then $\mathrm{End}_{\mathbb{P}^n,d}$ is affine. By Theorem 18, the quotient $D_{d\curvearrowright n} = \mathrm{End}_{\mathbb{P}^n,d} / \mathrm{PGL}_{n+1}$ exists as an affine variety. To show it is a geomet-

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ric quotient, we only need to check the orbits. If there is a non-closed orbit $\mathrm{PGL}_{n+1}.f$, then choose $g \in \overline{\mathrm{PGL}_{n+1}.f} \setminus \mathrm{PGL}_{n+1}.f$. We have $\dim \mathrm{PGL}_{n+1}.f > \dim \mathrm{PGL}_{n+1}.g$. Note that for any $h \in \mathrm{End}_{\mathbb{P}^n,d}(\Bbbk)$, we have $\dim \mathrm{PGL}_{n+1}.h + \dim \mathrm{Aut}_h = \dim \mathrm{PGL}_{n+1}$. Then $\dim \mathrm{Aut}_g > \dim \mathrm{Aut}_f \geq 0$. This contradicts Proposition 34.

Step 2. We construct the natural transformation $\eta : \mathcal{D}_{d\curvearrowright n} \to \mathbf{h}_D$ and show that it gives a bijection $\mathcal{D}_{d\curvearrowright n}(\Bbbk) \to D_{d\curvearrowright n}(\Bbbk)$.

Given $T/\mathbf{k}$ and $(f : \mathcal{X} \to \mathcal{X}) \in \mathcal{D}_{d\curvearrowright n}(T)$, we construct $T \to D$ as follows. Fix an étale covering $\{T_i \to T\}$ trivializing $\mathcal{X}$ by Lemma 39.

On T_i, let P be the scheme parameterizing isomorphisms between $\mathcal{X}_{T_i}$ and $\mathbb{P}^n \times T_i$ over T_i. It is a PGL_{n+1}-torsor P and we have a universal family $\mathcal{I} : \mathcal{X}_{T_i} \times_{T_i} P \to (\mathbb{P}^n \times T_i) \times_{T_i} P$. Then we get

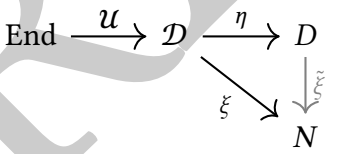
$$f' : (\mathbb{P}^n \times T_i) \times_{T_i} P \xrightarrow{\mathcal{I}^{-1}} \mathcal{X}_{T_i} \times_{T_i} P \xrightarrow{f\times \mathrm{id}} \mathcal{X}_{T_i} \times_{T_i} P \xrightarrow{\mathcal{I}} (\mathbb{P}^n \times T_i) \times_{T_i} P.$$

We easily see that this morphism is over $P = T_i \times_{T_i} P$ and has graph flat over P. Then it induces a morphism $P \to \mathrm{End}_{\mathbb{P}^n,d}$. This morphism is PGL_{n+1}-equivariant, where the action on the left is given by translation on P and on the right is given by conjugation. Taking the quotient, we get $T_i \to D_{d\curvearrowright n}$.

The above arguments are compatible with étale localization and base change. Then, gluing them, we get the morphism $T \to D$. And the map $\mathcal{D}_{d\curvearrowright n}(T) \to D_{d\curvearrowright n}(T)$ is natural and we get the natural transformation η. By the definition and construction, we have $\mathcal{D}_{d\curvearrowright n}(\Bbbk) = \mathrm{End}_{\mathbb{P}^n,d}(\Bbbk)/\mathrm{PGL}_{n+1}(\Bbbk) = D_{d\curvearrowright n}(\Bbbk)$ which coincides with $\eta(\Bbbk)$. Here the latter equality holds because the quotient is a geometric quotient.

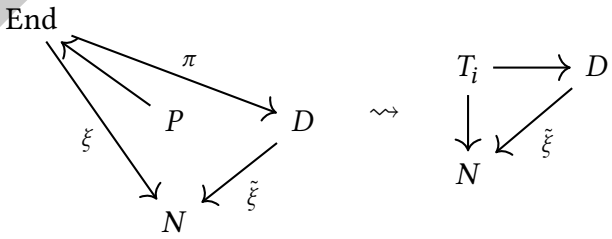
Step 3. Check the universal property of $\eta : \mathcal{D}_{d\curvearrowright n} \to D_{d\curvearrowright n}$ and hence finish the proof.

The universal family $\mathcal{U}$ on $\mathrm{End}_{\mathbb{P}^n,d}$ gives a morphism $\mathrm{End} \to \mathcal{D}$. Composing with η, we recover the quotient map $\mathrm{End} \to D$ since $\mathcal{U}$ is PGL_{n+1}-invariant. Then we have



We need to check the commutativity of the right triangle.

Fix $T/\mathbf{k}$ and $(f : \mathcal{X} \to \mathcal{X}/T) \in \mathcal{D}_{d\curvearrowright n}(T)$. Fix an étale covering $\{T_i \to T\}$ trivializing $\mathcal{X}$ by Lemma 39. On T_i, take P as in the previous step; we get the diagram on the left



and taking the quotient by PGL_{n+1}, we get the diagram on the right . On each T_i, $\xi_{T_i}(f) = \tilde{\xi} \circ \eta_{T_i}(f)$. Then they coincide on T. □

Lemma 39. Let $\pi : X \to S$ be a proper flat morphism with geometric fibers isomorphic to $\mathbb{P}^n$ and S noetherian. Then

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(a) if π has a section, then it is Zariski locally trivial;

(b) it is always étale locally trivial.

Proof. We show (a) and then (b) follows from (a) and [Sta26, Tag 055U].

Since π is flat with smooth geometric fibers, it is smooth. Its relative anticanonical bundle is relatively ample, so we may work locally on S with a projective morphism. Let $p : X^\vee \to S$ be the Hilbert scheme of hyperplanes in X over S as in Example 24 with universal divisor $\mathcal{U} \subset X \times_S X^\vee$. Consider the line bundle $\mathcal{L} = (\sigma \times \mathrm{id})^* \mathcal{O}_{X \times_S X^\vee}(\mathcal{U})$ on $X^\vee$, where $\sigma : S \to X$ is a section of X over S. Over $s \in S$, we have $\mathcal{L}|_{X_s^\vee} \cong \mathcal{O}(1)$ on $X_s^\vee \cong \mathbb{P}^n_{\kappa(s)}$. By cohomology and base change, $\mathcal{E} = p_* \mathcal{L}$ is locally free of rank $n+1$, and the evaluation map $p^* \mathcal{E} \to \mathcal{L}$ is surjective. Then the complete linear system gives $X^\vee \cong \mathbb{P}_S(\mathcal{E})$. Taking the relative dual projective space gives $X \cong \mathbb{P}_S(\mathcal{E}^\vee)$, so $X \to S$ is Zariski locally trivial. □

A good example is the degree two dynamics on the projective line; see Example 51.

Remark 40. The coarse moduli space $D_{d \curvearrowright n}$ is unirational since $\mathrm{End}_{\mathbb{P}^n, d}$ is rational and the quotient map $\mathrm{End}_{\mathbb{P}^n, d} \to D_{d \curvearrowright n}$ is surjective. Moreover, $D_{d \curvearrowright 1}$ is indeed rational for all $d \geq 2$; see [Lev11, Theorem 4.1].

Let us describe the distribution of dynamics with automorphism groups.

Theorem 41. Work over $\Bbbk$. Set $\Sigma := \{G < \mathrm{PGL}_{n+1}(\Bbbk) \mid \#G < \infty\} / \mathrm{PGL}_{n+1}(\Bbbk)$ with the conjugation action. The set Σ inherits a partial order from inclusion on $\{G < \mathrm{PGL}_{n+1}(\Bbbk) \mid \#G < \infty\}$. Associating every dynamical system $f \in D_{d \curvearrowright n}(\Bbbk)$ with the conjugation class of its automorphism group, we get a map

$$\Phi : D_{d \curvearrowright n}(\Bbbk) \to \Sigma := \{G < \mathrm{PGL}_{n+1}(\Bbbk) \mid \#G < \infty\} \Big/ \mathrm{PGL}_{n+1}(\Bbbk), \quad f \mapsto [\mathrm{Aut}_f(\Bbbk)].$$

(a) For any $[f_0] \in D_{d \curvearrowright n}(\Bbbk)$, there is an open neighborhood U of $[f_0]$ such that $\Phi([f]) \leq \Phi([f_0])$ for any $[f] \in U$.

(b) The map Φ can be extended to $D_{d \curvearrowright n}$ in the sense that for every scheme-theoretic point $\xi \in D_{d \curvearrowright n}$, there exists a nonempty open subset $U \subset \overline{\{\xi\}}$ such that Φ is constant on the closed points of U; then define $\Phi(\xi) = \Phi(U)$.

(c) For any x specializing ξ, we have the inclusion $\Phi(\xi) \leq \Phi(x)$.

Proof. First let us write down the partial order on Σ explicitly. For $[H], [G] \in \Sigma$, we say that $[H] \leq [G]$ if there exists $g \in \mathrm{PGL}_{n+1}(\Bbbk)$ such that $H \leq gGg^{-1}$. If we have $[H] \leq [G]$ and $[K] \leq [H]$, then there exist g, h with $H \leq gGg^{-1}$, $K \leq hHh^{-1}$. Then $K \leq hgG(hg)^{-1}$ and hence $[K] \leq [G]$. Reflexivity is clear. If $[H] \leq [G]$ and $[G] \leq [H]$, finiteness gives $\#H = \#G$, so H and G are conjugate. Thus this is a partial order.

Step 1. Proof of (a).

Given $f_0 \in \mathrm{End}_{\mathbb{P}^n, d}(\Bbbk)$, choose S as in Luna's étale slice theorem (Theorem 21). Let U be the image of $S / \mathrm{Aut}_{f_0} \to D_{d \curvearrowright n}$. We have a diagram

$$\begin{array}{ccccc} \mathrm{PGL}_{n+1} \times^{\mathrm{Aut}_{f_0}} S & \xrightarrow{\varphi} & \pi^{-1}(U) & \hookrightarrow & \mathrm{End}_{\mathbb{P}^n, d} \\ \downarrow p & & \downarrow & & \downarrow \pi \\ S / \mathrm{Aut}_{f_0} & \xrightarrow{\bar{\varphi}} & U & \hookrightarrow & D_{d \curvearrowright n} \end{array}$$

For any $f \in \pi^{-1}(U)(\Bbbk)$, choose $s \in (\mathrm{PGL}_{n+1} \times^{\mathrm{Aut}_{f_0}} S)(\Bbbk)$ mapping to f. The Cartesian square identi-

fies the fiber over $p(s)$ with the fiber over $[f]$ by a PGL_{n+1}-equivariant isomorphism. On geometric points, both fibers are single orbits. In particular, the equivariant bijection gives $G_s = \mathrm{Aut}_f(\Bbbk)$ as subgroups of $\mathrm{PGL}_{n+1}(\Bbbk)$, where G_s is the stabilizer of s. However, for $s = [a, h] \in (\mathrm{PGL}_{n+1} \times^{\mathrm{Aut}_{f_0}} S)(\Bbbk)$, if $g \in G_s$, then $[ga, h] = [a, h]$ and then there exists $k \in \mathrm{Aut}_{f_0}(\Bbbk)$ with $ga = ak^{-1}$, that is, $g \in a\,\mathrm{Aut}_{f_0}(\Bbbk)a^{-1}$. Hence $[\mathrm{Aut}_f(\Bbbk)] = [G_s] \leq [\mathrm{Aut}_{f_0}(\Bbbk)]$.

Step 2. Proof of (b) and (c).

For every $\xi \in D_{d\curvearrowright n}$, set $Z = \overline{\{\xi\}}$ and choose a closed point $y \in Z$ whose automorphism group has the smallest possible order among the closed points of Z. By (a), there is an open neighborhood U of y with $\Phi(z) \leq \Phi(y)$ for every closed point $z \in U$. For every closed point $z \in U \cap Z$, minimality gives $\Phi(z) = \Phi(y)$. Since Z is irreducible, $U \cap Z$ is dense in Z. Define $\Phi(\xi) = \Phi(y)$. Any two such dense open subsets intersect in a closed point, so this definition is independent of the choice. This proves (b).

For (c), fix $[H] \in \Sigma$. By (a), the closed points z satisfying $[H] \not\leq \Phi(z)$ form the closed points of an open subset $O_{[H]}$ of $D_{d\curvearrowright n}$: take the union of the neighborhoods in (a) of these points. Set $C_{[H]} = D_{d\curvearrowright n} \setminus O_{[H]}$. For any scheme point ξ, take a dense open subset of $\overline{\{\xi\}}$ as in (b). Since schemes of finite type over $\Bbbk$ are Jacobson, the closed points of this open subset are dense, and we obtain

$$\xi \in C_{[H]} \quad \Longleftrightarrow \quad [H] \leq \Phi(\xi).$$

Now take $[H] = \Phi(\xi)$. The closed subset $C_{[H]}$ contains ξ and hence every specialization x of ξ. Thus $\Phi(\xi) \leq \Phi(x)$, proving (c). □

Remark 42. The Theorem 41 can be explained as some semicontinuity of stabilizer groups under inclusion.

Corollary 43. There are only finitely many possible conjugacy classes of $\mathrm{Aut}_{[f]}(\Bbbk) < \mathrm{PGL}_{n+1}(\Bbbk)$ with $[f] \in \mathcal{D}_{d\curvearrowright n}$. In particular, there is a number $C_{n,d}$ depending only on d, n such that $\#\,\mathrm{Aut}_{[f]}(\Bbbk) < C_{n,d}$ for all $[f] \in \mathcal{D}_{d\curvearrowright n}(\Bbbk)$.

Remark 44. In his paper [Lev11], Levy provided a more concrete proof for the uniform bound on the size of automorphism groups. His proof depends on the explicit expressions of endomorphisms and automorphisms on $\mathbb{P}^n$.

Remark 45. In his paper [Lev11], Levy also proves that for general $[f] \in D_{d\curvearrowright n}(\Bbbk)$, $\mathrm{Aut}_f = \{*\}$ is the trivial group. By Theorem 41, we only need to find one dynamical system with trivial automorphism group.

4 Multipliers and example of degree two rational functions

In this subsection, we use the *multiplier spectra* to describe $\mathcal{D}_{2\curvearrowright 1}$. We temporarily use the notation $N_m = N_m(d \curvearrowright n) = ((d^m)^{n+1} - 1)/(d^m - 1)$ to denote the number of elements in the set $\mathrm{Per}_m(f)$ for a general $f \in \mathcal{D}_{d\curvearrowright n}(\Bbbk)$.

First we define the morphism to the set of periodic points.

Definition 46. Let $f : X \to X$ be a surjective endomorphism of a smooth projective variety and $m \geq 1$. Its fixed-point scheme $\mathrm{Fix}(f^m)$ is the scheme-theoretic intersection of the graph of f^m with the diagonal.

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Its *periodic points (of order dividing m)* and *preperiodic points* are the sets

$$\mathrm{Per}_m(f) := \{x \in X(\Bbbk) \mid f^m(x) = x\}, \quad \mathrm{Per}(f) := \bigcup_{m\geq 1} \mathrm{Per}_m(f), \quad \mathrm{Preper}(f) := \bigcup_{r\geq 0} f^{-r}(\mathrm{Per}(f)).$$

We say that f has *isolated periodic points* if $\mathrm{Fix}(f^m)$ is zero-dimensional for every $m \geq 1$. In this case, we also view $\mathrm{Per}_m(f)$ as a multiset by assigning each x the multiplicity $\mathrm{length}(\mathcal{O}_{\mathrm{Fix}(f^m),x})$ after base change to $\Bbbk$.

Note that $f \in \mathcal{D}_{d\curvearrowright n}(\Bbbk)$ always has isolated periodic points. Indeed, if f^m fixes a positive-dimensional subvariety Z pointwise, then restricting $(f^m)^*\mathcal{O}(1) \cong \mathcal{O}(d^m)$ to Z gives $\mathcal{O}_Z(1) \cong \mathcal{O}_Z(d^m)$, contradicting ampleness since $d \geq 2$.

Proposition 47. For any $m \geq 1$, the map

$$\mathrm{End}_{\mathbb{P}^n,d}(\Bbbk) \to \mathrm{Sym}^{N_m}\mathbb{P}^n(\Bbbk), \quad [f] \mapsto [\mathrm{Per}_m(f)]$$

comes from a morphism $\mathrm{Per}_m : \mathrm{End}_{d\curvearrowright n} \to \mathrm{Sym}^{N_m}\mathbb{P}^n$.

Proof. Let $\mathcal{U} \subset \mathbb{P}^n \times \mathbb{P}^n \times \mathrm{End}_{\mathbb{P}^n,d}$ be the universal family and $\Delta_{12} \subset \mathbb{P}^n \times \mathbb{P}^n \times \mathrm{End}_{\mathbb{P}^n,d}$ be the pullback of the diagonal on $\mathbb{P}^n \times \mathbb{P}^n$ along the projection. Let $F = \mathcal{U} \cap \Delta_{12}$ be the scheme-theoretic intersection. Set $H = \mathrm{End}_{\mathbb{P}^n,d}$. Since every fixed-point scheme is zero-dimensional, $F \to H$ is proper and quasi-finite, hence finite. The diagonal is locally defined by n equations, so F is locally defined by n equations in the smooth scheme $\mathcal{U} \cong \mathbb{P}^n \times H$. By finiteness, every irreducible component of F has dimension at most $\dim H$; by the number of equations, it has dimension at least $\dim H$. Hence these equations have codimension n and form a regular sequence. Thus F is Cohen-Macaulay; see [Sta26, Tag 00N6 and Tag 02JN]. Since H is smooth, $F \to H$ is flat by miracle flatness ([Sta26, Tag 00R4]). Its degree is constant since H is integral. For the power map $[x_0 : \cdots : x_n] \mapsto [x_0^d : \cdots : x_n^d]$, all fixed points are simple, and counting by the number of nonzero coordinates gives

$$\deg(F/H) = \sum_{r=1}^{n+1} \binom{n+1}{r} (d-1)^{r-1} = \frac{d^{n+1}-1}{d-1} = N_1.$$

Hence we get a morphism

$$\mathrm{End}_{\mathbb{P}^n,d} \to \mathrm{Hilb}^{N_1}\mathbb{P}^n \to \mathrm{Sym}^{N_1}\mathbb{P}^n, \quad f \mapsto F_f \mapsto [\mathrm{Fix}(f)],$$

where the first morphism comes from the flat family F and the second is the Hilbert–Chow morphism (see Example 23). The morphisms of periodic points can be obtained by

$$\mathrm{End}_{\mathbb{P}^n,d} \xrightarrow{f\mapsto(f,\cdots,f)} \mathrm{End}_{\mathbb{P}^n,d} \times \cdots \times \mathrm{End}_{\mathbb{P}^n,d} \xrightarrow{\text{composition}} \mathrm{End}_{\mathbb{P}^n,d^m} \xrightarrow{\mathrm{Fix}} \mathrm{Sym}^{N_m}\mathbb{P}^n.$$

We are done. □

Definition 48. Let $f \in \mathrm{End}_{\mathbb{P}^n,d}(\Bbbk)$. We define

$$\Lambda^m(f) := \bigcup_{p\in \mathrm{Per}_m(f)} \left\{ \text{eigenvalues of } (f^m)_* : T_p\mathbb{P}^n \to T_p\mathbb{P}^n \right\}.$$

Here both Per_m and the set of eigenvalues are multisets.

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We easily see that $\Lambda^m(f)$ depends only on $[f] \in \mathcal{D}_{d\curvearrowright n}$ and it takes values in $E_{nN_m}(\Bbbk) \cong \mathrm{Sym}^{nN_m} \mathbb{A}^1(\Bbbk) \cong \mathbb{A}^{nN_m}(\Bbbk)$, where $E_n = M_n/\mathrm{GL}_n$ as in Example 19 and Example 31.

Proposition 49. For any $m \geq 1$, the map

$$\mathcal{D}_{d\curvearrowright n}(\Bbbk) \to \mathbb{A}^{nN_m}(\Bbbk), \quad [f] \mapsto \Lambda^m(f)$$

indeed comes from a morphism $\Lambda^m : D_{d\curvearrowright n} \to E_{nN_m} \cong \mathbb{A}^{nN_m}$.

Proof. First consider $m = 1$. Set $H = \mathrm{End}_{\mathbb{P}^n,d}$, $\mathcal{X} = \mathbb{P}^n \times H$ and let $\mathcal{f} : \mathcal{X} \to \mathcal{X}$ be the universal endomorphism over H. Denote by $i : F \hookrightarrow \mathcal{X}$ its scheme-theoretic fixed locus and $p : F \to H$ the projection. As in the proof of Proposition 47, p is finite flat of degree N_1. By the definition of F, we have $\mathcal{f} \circ i = i$. Hence the relative differential restricts to an endomorphism

$$i^*(\mathcal{f}_*) : \mathcal{E} \to \mathcal{E}, \quad \mathcal{E} := i^*\mathcal{T}_{\mathcal{X}/H}.$$

Since p is finite flat, pushing forward the endomorphism to H, we get $A = p_*(i^*(\mathcal{f}_*)) : p_*\mathcal{E} \to p_*\mathcal{E}$ with $p_*\mathcal{E}$ locally free of rank nN_1. By Example 31, we get the morphism

$$a : H \to E_{nN_1}, \quad a = (\mathrm{Tr}\, A, \mathrm{Tr}\, A^2, \cdots, \mathrm{Tr}\, A^{nN_1}).$$

We check that $a(f)$ is indeed $\Lambda^1(f)$, including the multiplicities. At a geometric point f, filter each local algebra $\mathcal{O}_{F_f,x}$ by powers of its maximal ideal. On the associated graded pieces, A_f acts by the tangent map $(f_*)_x$ tensored with the identity. Hence, for every $j \geq 1$,

$$\mathrm{Tr}(A_f^j) = \sum_{x \in \mathrm{Supp}(F_f)} \mathrm{length}(\mathcal{O}_{F_f,x}) \,\mathrm{Tr}\big((f_*)_x^j\big).$$

Thus its eigenvalue multiset is exactly $\Lambda^1(f)$.

For general m, apply the same construction to $\mathrm{End}_{\mathbb{P}^n,d^m}$ and compose with the morphism $f \mapsto f^m$. Then we get a diagram

$$\begin{array}{ccccc} \mathrm{End}_{\mathbb{P}^n,d} & \xrightarrow{f \mapsto f^m} & \mathrm{End}_{\mathbb{P}^n,d^m} & \xrightarrow{a} & E_{nN_m} \\ \downarrow & & & \nearrow_{\exists! \Lambda^m} & \\ D_{d\curvearrowright n} & & & & \end{array}$$

Conjugation preserves the fixed-point multiplicities and conjugates the tangent maps, so the top composite is PGL_{n+1}-invariant. By the universal property of the categorical quotient $D_{d\curvearrowright n}$, it descends uniquely. Hence we get the desired gray morphism. □

Remark 50. The above arguments work for smooth X with a smooth Hilbert scheme of amplified endomorphisms since amplified dynamical systems have dense and isolated periodic points; see [Men23].

Example 51. For $n = 1$, $d = 2$, the morphism $\Lambda^1 : D_{2\curvearrowright 1} \to E_3 \cong \mathrm{Sym}^3 E_1 \cong \mathbb{A}^3$ in Proposition 49 is a closed embedding with image $\mathbb{A}^2$. Here we sketch a proof; for details, see [Sil12, Section 2.5].

Here on $\mathbb{A}^3$ we use the coordinates

$$\sigma_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad \sigma_2 = \sum_{i>j} \lambda_i \cdot \lambda_j, \quad \sigma_3 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3,$$

where λ_i are the three multipliers of $f \in D_{2\curvearrowright 1}(\Bbbk)$, counted with the fixed-point multiplicities. On the

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dense open subset where the three fixed points are simple, the fixed-point index formula (obtained from the residue theorem) gives

$$\frac{1}{1-\lambda_1}+\frac{1}{1-\lambda_2}+\frac{1}{1-\lambda_3}=1.$$

Clearing denominators gives $\sigma_1=\sigma_3+2$. Since the multiplier map is a morphism, this polynomial identity holds on all of $D_{2\curvearrowright 1}$. This will be the unique restriction.

Suppose that $\{\lambda_1,\lambda_2,\lambda_3\}\neq\{1,1,1\}$. Choose one multiplier $\lambda\neq 1$. Its fixed point is simple, so there is another distinct fixed point with multiplier μ. Putting these two points at $0,\infty$ and rescaling gives the normal form

$$f(z)=\frac{z^2+\lambda z}{\mu z+1},\qquad \lambda\mu\neq 1.$$

Conversely, this formula has degree two whenever $\lambda\mu\neq 1$, and the third multiplier is

$$\nu=\frac{2-\lambda-\mu}{1-\lambda\mu}.$$

For any triple satisfying $\lambda_1+\lambda_2+\lambda_3=\lambda_1\lambda_2\lambda_3+2$ other than $\{1,1,1\}$, we can choose $\lambda\neq 1$ and another multiplier μ with $\lambda\mu\neq 1$. Indeed, if both other multipliers equal $1/\lambda$, the relation gives $\lambda+1/\lambda=2$, contradicting $\lambda\neq 1$. The relation then determines the third multiplier by the above formula. Hence the normal form realizes every such triple and determines its conjugacy class.

If $\{\lambda_i\}=\{1,1,1\}$, there is only one fixed point: otherwise the same two-point normal form would have $\lambda\mu=1$ and degree less than two. Putting this fixed point at ∞, an easy translation and rescaling give $f(z)=z+1/z$. Thus this triple also determines a unique conjugacy class. Hence we obtain a morphism $D_{2\curvearrowright 1}\to V(\sigma_1-\sigma_3-2)\cong\mathbb{A}^2$ which is bijective on geometric points. It is quasi-finite, and it is birational since the characteristic is zero and both varieties are integral. Since the target is normal, Zariski's main theorem makes this morphism an open immersion. By surjectivity it is an isomorphism. Therefore Λ^1 is a closed embedding with the stated image.

Remark 52. Consider $L^m=(\Lambda^1,\cdots,\Lambda^m):D_{d\curvearrowright 1}\to\mathbb{A}^N$. McMullen shows that L^m is finite-to-one onto its image outside the Lattès family for all $d\geq 2$ and m sufficiently large; see [McM87]. Ji and Xie show it is generically injective for m sufficiently large; see [JX25].

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