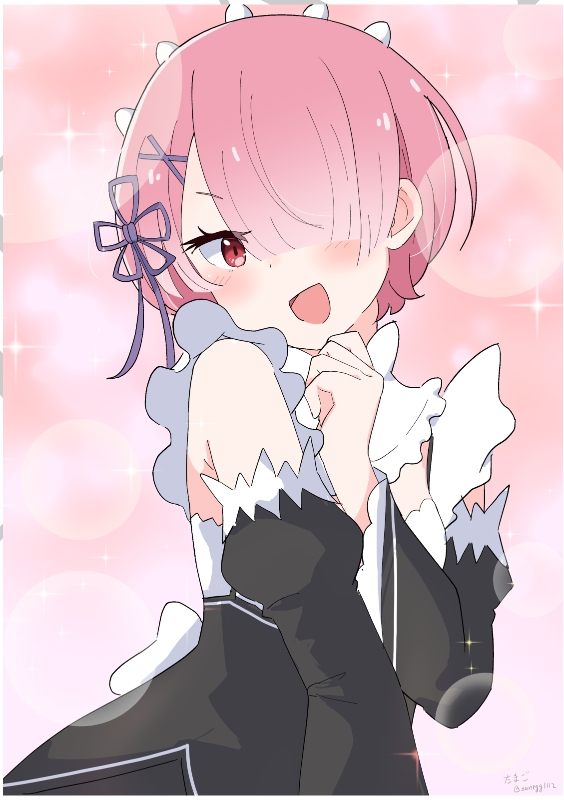
Notes on reading Xie's paper *Algebraic Dynamics and Recursive Inequalities*



“拉姆已经足够可爱了，再可爱下去，世界就要有危险了。”

Contents

1 Introduction

We work over an arbitrary field $\mathbf{k}$ without any restriction on the characteristic. A variety is always assumed to be geometrically integral.

Theorem A (= Theorem 3.7, ref. [Xie24, Theorem 3.7]). Let X be a projective variety of dimension d and $f : X \dashrightarrow X$ be a dominant rational map. For $i = 0, 1, \dots, d$ and $\varepsilon \in (0, 1)$, there exists $m_\varepsilon > 0$ such that for all $m \geq m_\varepsilon$,

$$L_{2m} - \varepsilon^m \mu_i^m L_m + \mu_i^m \mu_{i+1}^m L$$

is big.

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Theorem B (= Theorem 4.3, ref. [Xie24, Theorem 1.8])**.** Let S be an integral noetherian scheme, and $\pi : X \to S$ be a flat projective morphism. Let $f : X \dashrightarrow X$ be a dominant rational map over S. Then for every i, the function $s \mapsto \lambda_i(f_s)$ is lower semi-continuous.

Theorem C (= Theorem 5.9, ref. [Xie24, Theorem 1.12])**.** Let X be a projective variety and $f : X \dashrightarrow X$ be a dominant rational map. If f is cohomologically hyperbolic, then the set of periodic points of f is Zariski dense in X.

Theorem D (= Yang: , ref. [Xie24, Theorem 1.14])**.** Let X be a projective surface over $\overline{\mathbb{Q}}$ and $f : X \dashrightarrow X$ be a dominant rational map. Assume that $\mu_2(f) < 1$ or $\mu_1(f) = \mu_2(f)$. Then the Kawaguchi-Silverman conjecture holds for (X, f).

2 Preliminaries

2.1 Varieties and rational maps

Throughout this note, all schemes are of finite type and separated over a field or a noetherian integral scheme. A variety is a geometrically integral scheme of finite type and separated over a field. In other words, when base field is algebraically closed, a variety is an integral scheme of finite type and separated over the base field. And in general case, X is a variety if and only if its base change to the algebraic closure of the base field is a variety. We usually use $\mathbf{k}$ to denote an arbitrary field and $\Bbbk$ to denote an algebraically closed field.

When work over a non-algebraically closed field $\mathbf{k}$, we often consider the set $X(\Bbbk)$ of $\Bbbk$-points of a variety X over $\mathbf{k}$ rather than the scheme-theoretic points of X. Recall that for any field extension $\mathbf{k}'/\mathbf{k}$, the set of $\mathbf{k}'$-points of X is given by

$$X(\mathbf{k}') = \mathrm{Mor}_{\mathbf{k}}(\operatorname{Spec} \mathbf{k}', X) = \{x : \operatorname{Spec} \mathbf{k}' \to X \text{ over } \mathbf{k}\}.$$

A point $x \in X(\Bbbk)$ is determined by the image of x in X and the induced embedding $\kappa(x) \hookrightarrow \mathbf{k}'$ of the residue field $\kappa(x)$ of x into $\mathbf{k}'$. For a scheme-theoretic closed point $\xi \in X$, we will say that $\xi \in X(\mathbf{k}')$ if the residue field $\kappa(\xi)$ of ξ is a subfield of $\mathbf{k}'$, or equivalently, there exists $x \in X(\mathbf{k}')$ such that the image of x in X is ξ. Note that there may be at most $[\kappa(\xi) : \mathbf{k}]$ different points in $X(\mathbf{k}')$ with the same image ξ in X.

Proposition 2.1. Let X, Y be varieties over $\mathbf{k}$ and $f : X \to Y$ be a morphism. Then

(a) $X(\Bbbk)$ is canonically identified with the set of closed points of $X_{\Bbbk}$;

(b) $f : X(\Bbbk) \to Y(\Bbbk), x \mapsto f(x) = x \circ f$ coincides with the morphism $f_{\Bbbk} : X_{\Bbbk} \to Y_{\Bbbk}$ on the closed points under the above identification.

Remark 2.2. If we consider the set of scheme-theoretic points of X rather than $\Bbbk$-points, then the morphism $f : X \to Y$ may behave unlike our intuition. For example, let X be a variety over finite field $\mathbb{F}_q$ and consider the absolute Frobenius morphism Frob_q. Then Frob_q is the identity morphism on the set of scheme-theoretic points of X. But on $X(\overline{\mathbb{F}_q})$, it has many interesting dynamical properties.

Let X, Y be a variety over $\mathbf{k}$, and $f : X \dashrightarrow Y$ be a rational map. Recall that a rational map $f : X \dashrightarrow Y$ is a morphism defined on a dense open subset of X. The fundamental method to study a rational map is considering its graph. Let $\Gamma_f \subset X \times Y$ be the scheme-theoretic image of $(\mathrm{id}_U, f) : U \to X \times Y$, where $U \subset X$ is the dense open subset where f is defined. This is equivalent to taking the closure of set-theoretic image of (id_U, f) in $X \times Y$.

Definition 2.3. Let X be a variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ be a dominant rational map. We say that a point $x \in X(\Bbbk)$ is a *periodic point of f (of period r)* if f is defined at the orbit of x and $f^r(x) = x$ for some $r > 0$.
We say that a scheme-theoretic point $\xi \in X$ is a periodic point (of period r) of f if it is the image of a periodic point (of period r) of f in $X(\Bbbk)$.
We denote by $\mathrm{Per}_r(f)$ the set of periodic points of f of period r and by $\mathrm{Per}(f) = \bigcup_{r>0} \mathrm{Per}_r(f)$ the set of periodic points of f.
A periodic point $x \in X(\Bbbk)$ of f is called *isolated* if it(or its image) is isolated in the $\mathrm{Per}_r(f)$ for its some period r.

Proposition 2.4. Let X be a variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ be a dominant rational map. Then $\xi \in \mathrm{Per}_r(f)$ if and only if f is defined at ξ and ξ lies in set-theoretic image of $\pi_1 : \Gamma_{f^r} \cap \Delta_X \to X$ for some $r > 0$, where Γ_{f^r} is the graph of f^r and Δ_X is the diagonal of $X \times X$.

Proof. We have $\mathrm{Per}_r(f) \subset \pi_1(\Gamma_{f^r} \cap \Delta_X)$ since if $\xi \in \mathrm{Per}_r(f)$, then there exists $x \in X(\Bbbk)$ such that ξ is the image of x in X and $f^r(x) = x$. Hence $(x, x) \in (\Gamma_{f^r} \cap \Delta_X)(\Bbbk)$ and ξ is the image of $\pi_1(x, x) \in \pi_1(\Gamma_{f^r} \cap \Delta_X)(\Bbbk)$.

Conversely, if $\xi \in \pi_1(\Gamma_{f^r} \cap \Delta_X)$, then there exists a scheme-theoretic point $\zeta \in \Gamma_{f^r} \cap \Delta_X$ such that $\pi_1(\zeta) = \xi$. Pick any $y \in \Gamma_{f^r} \cap \Delta_X(\Bbbk)$ with image ζ in $\Gamma_{f^r} \cap \Delta_X$, then $\pi_1(y)$ has image ξ in X. Since f is defined at ξ, we have $\pi_1(y)$ is a periodic point of f of period r. Hence $\xi \in \mathrm{Per}_r(f)$. ☐

The graph Γ_f is a variety birational to X via π_1 and f can be recovered as the composition $\pi_2 \circ \pi_1^{-1}$, where $\pi_i : \Gamma_f \to X$ is the restriction of the i-th projection $X \times Y \to X$. Using the graph, we can define the pullback and pushforward of divisors by f as follows:

$$f^* D := \pi_{1*} \pi_2^* D, \quad f_* D := \pi_{2*} \pi_1^* D.$$

However, not like morphisms, the pullback and pushforward by rational maps are not functorial in general, which causes some technical difficulties in the study of rational maps. We will fix this issue by considering $\mathbf{b}$-cycles on birational models in Subsection 2.2.

Example 2.5. Consider the rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by $[x : y : z] \mapsto [1/x : 1/y : 1/z] = [yz : xz : xy]$. Then graph of f is given by

$$\Gamma_f = \{([x : y : z], [u : v : w]) \in \mathbb{P}^2 \times \mathbb{P}^2 : xu = yv = zw\}.$$

Consider the first projection $\pi_1 : \Gamma_f \to \mathbb{P}^2$ and $\mathbb{A}^2 \cong U \subset \mathbb{P}^2$ given by $z \neq 0$. Then

$$\begin{aligned} \pi_1^{-1}(U) &= \{([x : y : 1], [u : v : w]) \in \Gamma_f : xu = yv = w\} \\ &= \{([x : y : 1], [u : v]) \in U \times \mathbb{P}^1 : xu = yv\}, \end{aligned}$$

which is the blowup of U at the origin. By similar argument, Γ_f is the blowup of $\mathbb{P}^2$ at $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$ and π_1 is the blowup morphism.

We denote by L_x, L_y and L_z the coordinate lines in $\mathbb{P}^2$ given by $x = 0$, $y = 0$ and $z = 0$ respectively, and E_x, E_y and E_z the exceptional divisors in Γ_f over $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$ respectively. On Γ_f, the strict transform of L_x's and the exceptional divisors E_x's form a hexagon of rational curves. Taking $H = L_x$ on $\mathbb{P}^2$, then we have $p_2^* H = \widetilde{L_x} + E_y + E_z$. When we push forward by π_1, the strict transform $\widetilde{L_x}$ is contracted to a point, while E_y and E_z are mapped isomorphically to L_y and L_z respectively. Hence we have $f^* H = \pi_{1*}\pi_2^* H = L_y + L_z$. Then we get $(f^*)^2 H \sim f^* 2H \sim 4H$ while $(f^2)^* H = (\mathrm{id})^* H = H$.

We need to consider family of dominant rational maps.

Definition 2.6. Let S be a noetherian integral scheme. A *family of dominant rational maps* over S is the following data:

- a scheme X of finite type and flat over S such that every fiber X_s with $s \in S$ is a variety over $\kappa(s)$;
- an open dense subset $U \subseteq X$ and a morphism $f : U \to X$ over S such that for every $s \in S$, the restriction of f to the fiber U_s gives a dominant rational map on X_s.

We denote by $f : X \dashrightarrow X$ the family of dominant rational maps over S.

Given a family of dominant rational maps $f : X \dashrightarrow X$ over S, we can similarly define the graph $\Gamma_f \subset X \times_S X$ as the scheme-theoretic image (see [Stacks, Tag 01R5]) of the morphism $(i_U, f) : U \to X \times_S X$, where $i_U : U \to X$ is the inclusion morphism. As varieties, this is equivalent to taking the closure of set-theoretic image of (id_U, f) in $X \times_S X$. We have Γ_f is a scheme of finite type and separated over S which is birational to X via the first projection $\pi_1 : \Gamma_f \to X$.

Proposition 2.7. Let $f : X \dashrightarrow X$ be a family of dominant rational maps over S. Then for every $s \in S$, Γ_{f_s} is a closed subscheme of $(\Gamma_f)_s$.

Proof. The commutative diagram

$$\begin{tikzcd}
U_s \arrow[r] \arrow[d] & \Gamma_{f_s} \arrow[r, hook] \arrow[d, gray] & X_s \times_{\kappa(s)} X_s \arrow[d] \\
U \arrow[r] & \Gamma_f \arrow[r, hook] & X \times_S X
\end{tikzcd}$$

induces the gray arrow $\Gamma_{f_s} \to \Gamma_f$, and base changing by $s \to S$ gives the morphism $\Gamma_{f_s} \to (\Gamma_f)_s$. This morphism composing with the closed immersion $(\Gamma_f)_s \to X_s \times_{\kappa(s)} X_s$ yields the closed immersion $\Gamma_{f_s} \to X_s \times_{\kappa(s)} X_s$, so itself is a closed immersion. ☐

Remark 2.8. In general, we have no equality $(\Gamma_f)_s = \Gamma_{f_s}$. For an example, see Example 4.2

The following lemma is well-known. We will use it for lifting isolated periodic points from special fibers to the generic fiber.

Lemma 2.9. Let X be a noetherian scheme and $Y \subset X$ be a regular immersion. For any subscheme $Z \subset X$ and $x \in Y \cap Z$ (if $Y \cap Z \neq \varnothing$), we have

$$\dim_x (Y \cap Z) \geq \dim_x Y + \dim_x Z - \dim_x X.$$

Proof. Suppose that Y is locally defined by a regular sequence $f_1, \dots, f_r$ in the local ring $\mathcal{O}_{X,x}$. Then we have

$$\dim_x(Y \cap Z) = \dim \mathcal{O}_{Z,x}/(f_1, \dots, f_r) \geq \dim \mathcal{O}_{Z,x} - r = \dim_x Z - (\dim_x X - \dim_x Y)$$

by Krull's principal ideal theorem. □

We may need the following special case of the flattening theorem of Gruson-Raynaud.

Theorem 2.10 (Gruson-Raynaud, ref. [Stacks, Tag 0815])**.** Let S be a noetherian integral scheme and X, Y of finite type and separated scheme over S. Let $f : X \to Y$ be a morphism over S and $U \subseteq Y$ be an open subset such that $f^{-1}(U) \to U$ is flat. Then there exists a blowup $\pi : Y' \to Y$ with center in $Y \setminus U$ such that the induced morphism $f' : X' \to Y'$ is flat, where X' is the Zariski closure of $f^{-1}(U) \times_U \pi^{-1}(U)$ in $X \times_Y Y'$.

2.2 Intersection product on birational models

Let X be a variety over $\mathbf{k}$. All birational models of X form a directed system, we denote by $\mathfrak{X}$. Let $\mathrm{CH}^i(X)$ be the Chow group of cocycles of codimension i on X; see [Ful13, Chapter 17]. Since the pullback by rational maps is not functorial, we need to consider the inductive limit of the Chow groups $\mathrm{CH}^i(X')$ for all birational models $X' \in \mathfrak{X}$. A dominant rational map $f : X \dashrightarrow Y$ between varieties of the same dimension will induce a "morphism" $\mathfrak{X} \to \mathfrak{Y}$ between the directed systems of birational models, which allows us to define a functorial pullback.

Definition 2.11. We define

$$\mathrm{CH}^i(\mathfrak{X}) := \varinjlim_{\pi : X'' \to X' \in \mathfrak{X}} \left(\pi^* : \mathrm{CH}^i(X') \to \mathrm{CH}^i(X'')\right).$$

An element in $\mathrm{CH}^i(\mathfrak{X})$ is called a **b**-*cocycle* of degree i on X. It can be viewed as a pair $(X', \alpha' \in \mathrm{CH}^i(X'))$ modulo the equivalence relation generated by $(X_1, \alpha_1) \sim (X_2, \alpha_2)$ if there is a higher birational model $X'' \xrightarrow{\pi_i} X_i$ for $i = 1, 2$ such that $\pi_1^* \alpha_1 = \pi_2^* \alpha_2$. The model X' is called a *defining model* of α. For $\alpha \in \mathrm{CH}^i(X)$, we still denote by α the **b**-cocycle defined by α.

The intersection product $\mathrm{CH}^i(X') \otimes \mathrm{CH}^j(X') \to \mathrm{CH}^{i+j}(X')$ induces an intersection product

$$\mathrm{CH}^i(\mathfrak{X}) \otimes \mathrm{CH}^j(\mathfrak{X}) \to \mathrm{CH}^{i+j}(\mathfrak{X})$$

by taking the intersection product on a common defining model. It is well-defined since the pullback is a ring homomorphism. The intersection number

$$\mathrm{CH}^i(\mathfrak{X}) \otimes \mathrm{CH}^{d-i}(\mathfrak{X}) \to \mathbb{Z}$$

is also well-defined by the projection formula.

Let $f : X \dashrightarrow Y$ be a dominant rational map between varieties of the same dimension. Then for every $\alpha \in \mathrm{CH}^i(\mathfrak{Y})$, f induces a rational map $X \dashrightarrow Y'$ with Y' a defining model of α. We will denote this data by $f : \mathfrak{X} \to \mathfrak{Y}$. Then f induces a morphism $X' \to Y'$ for some birational model $X' \in \mathfrak{X}$ by taking graph. The pullback $f^*\alpha$ is a well-defined element in $\mathrm{CH}^i(X')$, and hence defines an element in

$\mathrm{CH}^i(\mathfrak{X})$. Easily check that the pullback $f^*\alpha$ is independent of the choice of the defining model Y' of α and the choice of the birational model X' of X. In this way, if $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ are two dominant rational maps between varieties of the same dimension, then we have $(g \circ f)^* = f^* \circ g^*$ as homomorphisms $\mathrm{CH}^i(\mathfrak{Z}) \to \mathrm{CH}^i(\mathfrak{X})$.

Let us focus on the case of divisors, i.e. $i = 1$. In this case, note that the notion of nef, effective, big and pseudo-effective divisors are preserved by pullback, so we can also define these notions for **b**-divisors on $\mathfrak{X}$. For ampleness, it is not preserved by pullback. But we can still define an ample **b**-divisor as a **b**-divisor who has an ample defining model.

Let $f : X \dashrightarrow Y$ be a dominant rational map between varieties of the same dimension. Then for every ample **b**-divisor L on Y, f^*L will be also an ample **b**-divisor on X. Indeed, we may assume that Y is a defining model of L and L is an ample divisor on Y. Let X' be the graph of f and $\pi_1 : X' \to X$ and $f' : X' \to Y$ be the projections. Then f' is a generically finite morphism. Taking the Stein factorization of f', we get a factorization $X' \xrightarrow{\pi'} X'' \xrightarrow{f''} Y$ with π' birational and f'' finite. Then f''^*L is an ample divisor on X'' since f'' is finite. This gives an ample defining model X'' of f^*L.

Let $Z \subset X$ be a closed subvariety of dimension i and $\alpha \in \mathrm{CH}^i(\mathfrak{X})$. We can take a defining model X' of α which is higher than X by $\pi : X' \to X$. Suppose that Z is not contained in the indeterminacy locus of π^{-1}, then we can define the intersection number

$$\alpha \cdot Z := \alpha_{X'} \cdot \tilde{Z},$$

where $\tilde{Z}$ is the strict transform of Z on X'. Under suitable assumptions making the intersection numbers defined, we still have the projection formula for pullback by rational maps.

Proposition 2.12. Let $f : X \dashrightarrow Y$ be a dominant rational map between spaces of the same dimension, $\alpha \in \mathrm{CH}^i(Y)$ and $Z \subset X$ be a closed subvariety of dimension i. Suppose that Z is not contained in the indeterminacy locus of f, then the intersection number $f^*\alpha \cdot Z$ is defined and we have

$$f^*\alpha \cdot Z = \alpha \cdot f_* Z,$$

where $f_*Z := [\mathscr{K}(Z) : \mathscr{K}(\overline{f(Z)})] \cdot \overline{f(Z)}$.

Proof. Let X' be the graph of f and $\pi_1 : X' \to X$ and $f' : X' \to Y$ be the projections. Note that the indeterminacy locus of f is the same as the indeterminacy locus of π_1^{-1}, so Z is not contained in the indeterminacy locus of π_1^{-1}. By definition, we have X' is a defining model of $f^*\alpha$ since $f' : X' \to Y$ is a morphism. Let $\tilde{Z}$ be the strict transform of Z on X', then we have

$$f^*\alpha \cdot Z = \alpha_{X'} \cdot \tilde{Z} = f'^*\alpha \cdot \tilde{Z} = \alpha \cdot f'_*\tilde{Z} = \alpha \cdot f_* Z.$$

□

2.3 Siu's inequality and dynamical degrees

First we list some forms of Siu's inequality which will be used in the proof of the main theorem also in the proof of the properties of dynamical degrees.

Then original version of Siu's inequality is the following.

Theorem 2.13 (Siu's criterion, cf. [Laz04, Theorem 2.2.15])**.** Let X be a projective variety of dimension d and L, M be ample divisors on X. Then

$$M - \varepsilon \frac{1}{d} \frac{M^d}{L \cdot M^{d-1}} L$$

is big for every $\varepsilon \in (0, 1)$.

For higher codimensional cocycles, we have the following generalization of Siu's inequality. Note that this is a little weaker than the original version of Siu's inequality since bigness is replaced by the weaker condition that the intersection number is positive.

Theorem 2.14 (Generalized Siu's inequality, ref. [JL23])**.** Let X be a projective variety of dimension d and $\alpha \in \mathrm{CH}^k(X)$, $\beta \in \mathrm{CH}^{d-k}(X)$ be intersections of nef divisors on X and H a nef divisor on X. Then we have

$$(\alpha \cdot \beta) \cdot H^d \le \binom{d}{k} (\alpha \cdot H^{d-k})(\beta \cdot H^k).$$

By restricting the above inequality to hyperplane sections and taking limit to nef divisors, we can get the following version.

Corollary 2.15. Let X be a projective variety of dimension d and $\alpha \in \mathrm{CH}^k(X)$, $\beta \in \mathrm{CH}^l(X)$ and $\eta \in \mathrm{CH}^{d-k-l}(X)$ be intersections of nef divisors on X and H be a nef divisor on X. Then we have

$$(\alpha \cdot \beta \cdot \eta) \cdot (H^{k+l} \cdot \eta) \le \binom{k+l}{k} (\alpha \cdot H^l \cdot \eta)(\beta \cdot H^k \cdot \eta).$$

Now we can show the existence of dynamical degrees and their log-concavity.

Proposition 2.16. Let $f : X \dashrightarrow X$ be a rational map on a projective variety X of dimension d. Then the limit

$$\lambda_k(f) := \lim_{n \to \infty} \left(\deg_{k,H}(f^n) \right)^{1/n}$$

exists and is independent of the choice of the ample divisor H.

Proof. First

Yang: ☐

Proposition 2.17. The sequence of dynamical degrees $\lambda_k(f)$ is log-concave, i.e.

$$\lambda_k(f)^2 \ge \lambda_{k-1}(f) \lambda_{k+1}(f)$$

for every $1 \le k \le d-1$. Or equivalently, the sequence of Lyapunov exponents $\mu_k(f)$ is non-increasing, i.e.

$$\mu_k(f) \ge \mu_{k+1}(f)$$

for every $1 \le k \le d-1$.

Proof. Yang: ☐

2.4 Semi-continuous functions on noetherian schemes

Let S be an integral noetherian scheme, and $\theta : S \to \mathbb{R}$ be a function. We say that f is *lower semi-continuous* if for every $r \in \mathbb{R}$, the set $\{s \in S : f(s) > r\}$ is open in S.

Remark 2.18. The limit of a sequence of lower semi-continuous functions may not be lower semi-continuous. Yang:

Lemma 2.19. A function $\theta : S \to \mathbb{R}$ is lower semi-continuous if and only if the following condition holds:

(a) for every $\eta \in S$ and every specialization $\xi \in \overline{\{\eta\}}$, we have $\theta(\xi) \le \theta(\eta)$;

(b) for every $\eta \in S$ and $a < \theta(\eta)$, there is an open subset $U \subseteq \overline{\{\eta\}}$ such that $\theta(\xi) > a$ for every $\xi \in U$.

Proof. Yang: $\square$

Lemma 2.20. Let $\theta : S \to \mathbb{R}$ be a lower semi-continuous function. Then

(a) θ is continuous with respect to the constructible topology on S;

(b) if θ is discrete, then its image is finite;

(c) let $\eta \in S$ be the generic point, then $\lim_{s \to \eta} \theta(s) = \theta(\eta)$.

Proof. Yang: $\square$

2.5 The Frobenius

In this subsection, we review some fundamental properties of the Frobenius morphism, which may be trivial for experts. But for me, they are not so trivial at least at the beginning of this note.

Definition 2.21. Let S be a scheme over $\mathbb{F}_p$. Then the *absolute Frobenius* morphism $\mathrm{Frob}_{S,p} : S \to S$ is the morphism defined locally by the p-th power map on the structure sheaf $\mathcal{O}_S$.

Proposition 2.22. Let S, T be schemes over $\mathbb{F}_p$ and $f : S \to T$ be a morphism. Then f is commutative with the absolute Frobenius Frob_p.

Definition 2.23. Let S be a scheme over $\mathbb{F}_p$ and X be a scheme over S. We denote by $X^{(p)}$ the base change of X along the absolute Frobenius $\mathrm{Frob}_{S,p}$, i.e. $X^{(p/S)} = X \times_{S,\mathrm{Frob}_{S,p}} S$, called the *Frobenius twist* of X. It is equipped with an S-scheme structure given by the second projection $X^{(p/S)} \to S$. The absolute Frobenius $\mathrm{Frob}_{S,p} : S \to S$ induces an S-morphism $\mathrm{id}_X \times \mathrm{Frob}_{S,p} : X \to X^{(p/S)}$, called the *relative Frobenius* morphism of X over S and denoted by $\mathsf{Frob}_{X/S,p}$. In a diagram, we have

$$\begin{tikzcd}
X \arrow[dr, "\mathsf{Frob}_{X/S,p}"] \arrow[drr, bend left, "\mathsf{Frob}_{X,p}"] \arrow[ddr, bend right] & & \\
& X^{(p/S)} \arrow[r] \arrow[d] & X \arrow[d] \\
& S \arrow[r, "\mathsf{Frob}_{S,p}"] & S.
\end{tikzcd}$$

Remark 2.24. We often omit the subscript S on the Frobenius twist when there is no confusion, i.e. $X^{(p)}$ for $X^{(p/S)}$. We also often omit the subscript X on the Frobenius morphism when there is no confusion, i.e. Frob_p for $\mathsf{Frob}_{X,p}$ and $\mathsf{Frob}_{/S,p}$ for $\mathsf{Frob}_{X/S,p}$. And we write Frob_{p^n} (resp. $\mathsf{Frob}_{/S,p^n}$) for the n-th iterate of Frob_p (resp. $\mathsf{Frob}_{/S,p}$).

Example 2.25. Let $S = \operatorname{Spec} \Bbbk$ be the spectrum of an algebraically closed field of characteristic p and $X \subset \mathbb{A}^n_{\Bbbk}$ be an affine variety defined by $f_1, \dots, f_m \in \Bbbk[t_1, \dots, t_n]$. Then the Frobenius twist $X^{(p)}$ is given by $X^{(p)} = \operatorname{Spec} \Bbbk[t_1, \dots, t_n]/(f_1^{(p)}, \dots, f_m^{(p)})$, where $f_i^{(p)}$ is obtained by applying the p-th power map to the coefficients of f_i. The projection $X^{(p)} \to X$ is given by $(a_1, \dots, a_n) \mapsto (a_1^{1/p}, \dots, a_n^{1/p})$, while the relative Frobenius $\mathsf{Frob}_{/S,p} : X \to X^{(p)}$ is given by $(a_1, \dots, a_n) \mapsto (a_1^p, \dots, a_n^p)$.

Relative Frobenius as endomorphisms when defined over finite fields The most interesting case for us is when $S = \operatorname{Spec} \mathbb{F}_q$ or $S = \operatorname{Spec} \overline{\mathbb{F}_q}$ and X is a variety over S. First consider the case when $S = \operatorname{Spec} \mathbb{F}_q$. Then the absolute Frobenius $\mathsf{Frob}_q : \operatorname{Spec} \mathbb{F}_q \to \operatorname{Spec} \mathbb{F}_q$ is the identity morphism, so the Frobenius twist $X^{(q)}$ is just X and the relative Frobenius $\mathsf{Frob}_{/S,q}$ is just the absolute Frobenius Frob_q.

More generally, let $\Bbbk$ be an arbitrary algebraically closed field of characteristic p and X be a variety over $\Bbbk$. Suppose that X is defined over some finite field $\mathbb{F}_q$. That is, there exists a variety X_0 over $\mathbb{F}_q$ and an isomorphism $X \cong X_0 \times_{\mathbb{F}_q} \Bbbk$. Then we have diagrams

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, red] \arrow[ddr] & \operatorname{Spec} \Bbbk \arrow[dr, "\mathsf{Frob}_q"] \arrow[ddr] & \\
& X \arrow[r] \arrow[d] & \operatorname{Spec} \Bbbk \arrow[d] \\
& X_0 \arrow[r] & \operatorname{Spec} \mathbb{F}_q
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
X \arrow[dr, blue] \arrow[drr, red] \arrow[ddr] & & \\
& X^{(q)} \arrow[r] \arrow[d] & X \arrow[d] \\
& \operatorname{Spec} \Bbbk \arrow[r, "\mathsf{Frob}_q"] & \operatorname{Spec} \Bbbk.
\end{tikzcd}$$

In particular, this gives an isomorphism $X^{(q)} \cong X$ (the blue arrow) over $\Bbbk$. In general, for a variety X over $\Bbbk$, we still have an isomorphism $X^{(q)} \cong X$ as schemes but not as $\Bbbk$-schemes. Using this $\Bbbk$-scheme isomorphism, the relative Frobenius $\mathsf{Frob}_{/\Bbbk,q} : X \to X^{(q)}$ can be viewed as a morphism $X \to X$, which is different from the absolute Frobenius $\mathsf{Frob}_q : X \to X$. Another more direct way to get the relative Frobenius is to take the product

$$\mathsf{Frob}_{/\Bbbk,q} = (\mathsf{Frob}_q)_{\Bbbk} = \mathsf{Frob}_{X_0/\mathbb{F}_q,q} \times \mathrm{id}_{\Bbbk} : X_0 \times_{\mathbb{F}_q} \Bbbk \to X_0 \times_{\mathbb{F}_q} \Bbbk.$$

This is the construction we will use in the following.

If $\Bbbk = \overline{\mathbb{F}_q}$, then every variety over $\Bbbk$ is defined over some finite field, so the above construction applies to every variety over $\Bbbk$.

Example 2.26 (Graph of Frobenius on $\mathbb{A}^1_{\mathbb{F}_p}$)**.** Let $X = \mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec}\mathbb{F}_p[t]$ be the affine line over $\mathbb{F}_p$. We consider the absolute Frobenius $F = \mathsf{Frob}_p : X \to X$ given by $\mathbb{F}_p[t] \to \mathbb{F}_p[t], f \mapsto f^p$. Note that even though F is the identity on the underlying topological space of X, the graph Γ_F of F is not the diagonal in $X \times X$.

Since $(\mathsf{id}, F) : X \to X \times X$ is given by $\mathbb{F}_p[t] \otimes_{\mathbb{F}_p} \mathbb{F}_p[t] \to \mathbb{F}_p[s], f(t) \otimes g(s) \mapsto f(t)g(t^p)$, the graph Γ_F is given by $\operatorname{Spec}\mathbb{F}_p[s,t]/(s - t^p)$. This is different from the diagonal Δ_X given by $\operatorname{Spec}\mathbb{F}_p[s,t]/(s-t)$.

The reason is that the underlying space $|X \times_{\mathbb{F}_p} X|$ of the fiber product $X \times_{\mathbb{F}_p} X$ is not the same as the product $|X| \times |X|$ of the underlying spaces. For example, let $\alpha \in \overline{\mathbb{F}_p} \setminus \mathbb{F}_p$ with minimal polynomial $f(t) \in \mathbb{F}_p[t]$. This defines a closed point $\mathfrak{m}_\alpha = (f(t)) \in |X|$. But there are several closed points in $|X \times_{\mathbb{F}_p} X|$ mapping to $\mathfrak{m}_\alpha$ via both projections. For example, I claim that the ideal $\mathfrak{m}_n = (f(t), s - t^{p^n})$ defines a closed point in $|X \times_{\mathbb{F}_p} X|$ mapping to $\mathfrak{m}_\alpha$ via both projections for every $n \geq 0$. Before proving the claim, note that $\mathfrak{m}_0 \neq \mathfrak{m}_1$. Otherwise, we would have $t - t^p \in \mathfrak{m}_0 \cap \mathbb{F}_p[t] = (f(t))$, which is impossible since $f(t)$ is irreducible of degree at least 2 and $t - t^p$ factors as $t(t-1)(t-2)\cdots(t-(p-1))$.

We denote by $\phi_1 : \mathbb{F}_p[t] \to \mathbb{F}_p[t,s]$ and $\phi_2 : \mathbb{F}_p[s] \to \mathbb{F}_p[t,s]$ the homomorphisms corresponding to the two projections $X \times_{\mathbb{F}_p} X \to X$. Then we have $\pi_i(\mathfrak{m}_n) = \phi_i^{-1}(\mathfrak{m}_n) = \ker(\mathbb{F}_p[t] \xrightarrow{\phi_i} \mathbb{F}_p[t,s]/\mathfrak{m}_n)$. Factor $\mathbb{F}_p[t,s] \to \mathbb{F}_p[t,s]/\mathfrak{m}_n$ as $\mathbb{F}_p[t,s] \to \mathbb{F}_p[t] \to \mathbb{F}_p[t]/(f(t))$, where the first map is given by $s \mapsto t^{p^n}$ and the second map is the quotient map. Then $\pi_1(\mathfrak{m}_n) = (f(t)) = \mathfrak{m}_\alpha$. For every $g(s) \in \mathbb{F}_p[s]$, regard $g(s)$ as a polynomial in $\mathbb{F}_p(t)[s]$, we have $s - t^{p^n} \mid g(s) - g(t^{p^n})$ in $\mathbb{F}_p(t)[s]$. Then there exists $h(t,s) \in \mathbb{F}_p[t,s]$ such that $g(s) - g(t^{p^n}) = h(t,s)(s - t^{p^n})$. In particular, $g(s) \in \mathfrak{m}_n$ if and only if $g(t^{p^n}) = g(t)^{p^n} \in \mathfrak{m}_n$, and the latter is equivalent to $g(t) \in (f(t)) = \mathfrak{m}_\alpha$ since $\mathfrak{m}_\alpha$ is a prime ideal. Hence $\pi_2(\mathfrak{m}_n) = (f(s)) = \mathfrak{m}_\alpha$ as well.

Proposition 2.27. Let X be a variety over $\mathbb{F}_q$. Then for every closed point $x \in X(\overline{\mathbb{F}_q})$, $x \in X(\mathbb{F}_{q^n})$ if and only if $\mathsf{Frob}_q^n(x) = x$. In particular, every closed point of X is a periodic point of the Frobenius Frob_q.

Proof. Let x be given by a morphism $x : \operatorname{Spec}\mathbf{k} \to X$ with $\mathbf{k}$ the residue field of x. Note that $\mathsf{Frob}_q^n(x)$ is given by the diagram

$$\begin{tikzcd}
\operatorname{Spec}\mathbf{k} \arrow[r, "x"] \arrow[d, "\mathsf{Frob}_q^n"'] \arrow[dr, "\mathsf{Frob}_q^n(x)"] & X \arrow[d, "\mathsf{Frob}_q^n"] \\
\operatorname{Spec}\mathbf{k} \arrow[r, "x"] & X.
\end{tikzcd}$$

Hence $\mathsf{Frob}_q^n(x) = x$ if and only if $\mathsf{Frob}_q^n : \operatorname{Spec}\mathbf{k} \to \operatorname{Spec}\mathbf{k}$ is the identity morphism, which is equivalent to $\mathbf{k} \subset \mathbb{F}_{q^n}$. ☐

Proposition 2.28. Let $f : X \to Y$ be a morphism of varieties over $\mathbf{k}$. Suppose that f, X, Y are defined over some finite field $\mathbb{F}_q$. Then f is commutative with the relative Frobenius $\mathsf{Frob}_{/\mathbf{k},q}$.

Proof. Choose $f_0 : X_0 \to Y_0$ defined over $\mathbb{F}_q$ such that $f = f_0 \times_{\mathbb{F}_q} \mathbf{k}$. Since f_0 is commutative with the absolute Frobenius Frob_q, the base change $f = f_0 \times_{\mathbb{F}_q} \mathbf{k}$ is also commutative with the relative Frobenius $\mathsf{Frob}_{/\mathbf{k},q} = (\mathsf{Frob}_q)_{\mathbf{k}}$. ☐

3 Estimation and Inequalities

In this section, fix a field $\mathbf{k}$, X a projective variety of dimension d over $\mathbf{k}$, L an ample line bundle on X and $f : X \dashrightarrow X$ a dominant rational self-map. Set $\lambda_i = \lambda_i(f)$ and $\mu_i = \mu_i(f)$ for every $i = 0, 1, \dots, d$. Set $\mu_{d+1} = 0$ for convenience.

3.1 Sequence with exponential growth

In this subsection, we will introduce some elementary notations and lemmas about sequences with exponential growth.

Definition 3.1. We say that a sequence of positive numbers a_n has *exponential growth* if $\lim_{n\to\infty} a_n^{1/n}$ exists and is positive.

Definition 3.2. Let a_n and b_n be two sequences of exponential growth. We write $a_n \preccurlyeq_e b_n$ (resp. $a_n \prec_e b_n$) if $\limsup_{n\to\infty} a_n/b_n \le 1$ (resp. < 1), and write $a_n \asymp_e b_n$ if $a_n \preccurlyeq_e b_n$ and $a_n \succcurlyeq_e b_n$.

Lemma 3.3. Suppose that $a_n \asymp_e a^n, b_n \asymp_e b^n$, then

(a) $a_n b_n \asymp_e (ab)^n$;

(b) $a_n/b_n \asymp_e (a/b)^n$;

(c) if $a_n \succ_e b_n$, then $a_n > b_n$ for every sufficiently large n;

(d) if $a \ge b$, then $a_n + b_n \asymp_e a^n$;

(e) if $a_n \ge b_n$ for every n and $a > b$, then $a_n - b_n \asymp_e a^n$.

Proof. □

Lemma 3.4. Let x_n be a sequence of positive numbers and $a > b > 0$ be some numbers. Suppose that

$$x_{n+2} + abx_n \ge (a+b)x_{n+1}, \quad \forall n \ge 0$$

and $x_1 > bx_0$. Then we have $x_n \succcurlyeq_e a^n$.

Proof. The inequality can be reformulated as

$$x_{n+2} - bx_{n+1} \ge a(x_{n+1} - bx_n).$$

Set $y_n := x_{n+1} - bx_n$. Then we have $y_{n+1} \ge ay_n$ for every $n \ge 0$ and $y_0 = x_1 - bx_0 > 0$. Hence, $y_n \ge a^n y_0 > 0$ for every $n \ge 0$. This implies $x_{n+1} = y_n + bx_n \ge y_n \succcurlyeq_e a^n$. □

3.2 Inequalities

Set

$$L_m := (f^m)^* L, \quad M_m(t) := L_{2m} + t^m L.$$

Let Z be a subvariety of X (or more generally, a cycle in $\mathrm{CH}^{\bullet}(X)$), set

$$\Sigma(t;Z)_m := \frac{1}{\dim Z}\frac{(M(t)_m|_Z)^{\dim Z}}{(M(t)_m|_Z)^{\dim Z-1}\cdot L_m|_Z} = \frac{1}{\dim Z}\frac{M(t)_m^{\dim Z}\cdot[Z]}{M(t)_m^{\dim Z-1}\cdot L_m\cdot[Z]}.$$

When $Z = X$, we simply write $\Sigma(t)_m$ for $\Sigma(t;X)_m$. By Siu's criterion (Theorem 2.14), we have $\big(M(t)_m - aL_m\big)|_Z$ is big on Z provided $\Sigma(t)_m > a$.

On the denominator of $\Sigma(t)_m$, there is some item of the form $L_{2m}^i\cdot L_m\cdot L^{d-1-i}$. This reminds us to estimate the intersection numbers of the form

$$L_{m_1}^{r_1}\cdot L_{m_2}^{r_2}\cdots L_{m_s}^{r_s},$$

where $m_1\geq m_2\geq\cdots\geq m_s$ and $r_1+r_2+\cdots+r_s = d$. In [Xie24], such intersection numbers are called *mixed degrees*.

Proposition 3.5. Let $r_1+\cdots+r_s = d$ be a partition of d into positive integers, and $R_i := r_1+\cdots+r_i$. Then for every $\mathbf{m} = (m_1,\dots,m_s)\in\mathbb{Z}_{\geq 0}^s$ with $m_1\geq m_2\geq\cdots\geq m_s$, we have

$$E(\mathbf{m})\cdot\prod_i \deg_{R_i}(f^{\Delta m_i}) \leq (L_{m_1}^{r_1}\cdot L_{m_2}^{r_2}\cdots L_{m_s}^{r_s}) \leq C_{\mathrm{mix}}\cdot\prod_i \deg_{R_i}(f^{\Delta m_i}),$$

where $\Delta m_i := m_i - m_{i+1}$ with $m_{s+1} := 0$ and

- C_{mix} is some constant depending only on (X,L);
- $E(\mathbf{m})$ is some error term satisfying

$$E(\mathbf{m}) \asymp_e 1 \quad \text{if } \mathbf{m} = (k_1 m,\dots,k_s m) \text{ for some } k_i\in\mathbb{Z}_{>0} \text{ and } m\to\infty.$$

In particular, if $\mathbf{m} = (k_1 m,\dots,k_d m)$ for some $k_1\geq k_2\geq\cdots\geq k_d\in\mathbb{Z}_{>0}$, then we have

$$(L_{m_1}\cdot L_{m_2}\cdots L_{m_d}) \asymp_e \prod_i \lambda_i^{\Delta m_i} = \mu_1^{m_1}\cdot\mu_2^{m_2}\cdots\mu_d^{m_d} \qquad \text{as } m\to\infty.$$

Proof.

Step 1. $(L_{m_1}^{r_1}\cdot L_{m_2}^{r_2}\cdots L_{m_s}^{r_s}) \leq C_{\mathrm{mix}}\prod_i \deg_{R_i}(f^{\Delta m_i})$.

By Siu's inequality, we have

$$(L_{m_2}^d)(L_{m_1}^{r_1}\cdot L_{m_2}^{r_2}\cdots L_{m_s}^{r_s}) \leq \binom{d}{r_1}(L_{m_1}^{r_1}\cdot L_{m_2}^{d-r_1})(L_{m_2}^{R_2}\cdots L_{m_s}^{r_s}).$$

By the projection formula, we have

$$\begin{aligned} L_{m_2}^d &= \deg(f^{m_2})L^d, \\ L_{m_1}^{r_1}\cdot L_{m_2}^{d-r_1} &= \deg(f^{m_1})L_{m_1-m_2}^{r_1}\cdot L^{d-r_1} = \deg(f^{m_2})\deg_{r_1}(f^{m_1-m_2}) = \deg(f^{m_2})\deg_{R_1}(f^{\Delta m_1}). \end{aligned}$$

Hence

$$(L_{m_1}^{r_1}\cdot L_{m_2}^{r_2}\cdots L_{m_s}^{r_s}) \leq \frac{1}{L^d}\binom{d}{r_1}\deg_{R_1}(f^{\Delta m_1})(L_{m_2}^{R_2}\cdots L_{m_s}^{r_s}).$$

Inducting on s, we can find a constant C depending only on (X,L) and r_i's such that

$$(L_{m_1}^{r_1}\cdot L_{m_2}^{r_2}\cdots L_{m_s}^{r_s}) \leq C\prod_i \deg_{R_i}(f^{\Delta m_i}).$$

Since there are only finitely many partitions of d, we can find a uniform constant C_{mix} depending only on (X, L).

Step 2. $E(\mathbf{m})\prod_i \deg_{R_i}(f^{\Delta m_i}) \le (L_{m_1}^{r_1}\cdot L_{m_2}^{r_2}\cdots L_{m_s}^{r_s})$.

By Siu's inequality, we have

$$(L_{m_2}^{r_1+r_2}\cdot\eta)(L_{m_1}^{r_1}L_{m_3}^{r_2}\cdot\eta) \le \binom{R_2}{r_1}(L_{m_1}^{r_1}L_{m_2}^{r_2}\cdot\eta)(L_{m_2}^{r_1}L_{m_3}^{r_2}\cdot\eta), \quad \eta = L_{m_3}^{r_3}\cdots L_{m_s}^{r_s}.$$

That is,

$$(L_{m_1}^{r_1}\cdots L_{m_s}^{r_s}) \ge \frac{1}{C_{R_2}^{r_1}}\frac{(L_{m_2}^{r_1+r_2}L_{m_3}^{r_3}\cdots L_{m_s}^{r_s})(L_{m_1}^{r_1}L_{m_3}^{r_2+r_3}\cdots L_{m_s}^{r_s})}{(L_{m_2}^{r_1}L_{m_3}^{r_2+r_3}\cdots L_{m_s}^{r_s})}.$$

By Step 1 and inducting on s, we may assume that

$$(L_{m_2}^{r_1+r_2}L_{m_3}^{r_3}\cdots L_{m_s}^{r_s}) \ge E_1(\mathbf{m})\deg_{R_2}(f^{\Delta m_2})\cdot\prod_{i=3}^{s}\deg_{R_i}(f^{\Delta m_i}),$$

$$(L_{m_1}^{r_1}L_{m_3}^{r_2+r_3}\cdots L_{m_s}^{r_s}) \ge E_2(\mathbf{m})\deg_{R_1}(f^{\Delta m_1+\Delta m_2})\cdot\prod_{i=3}^{s}\deg_{R_i}(f^{\Delta m_i}),$$

$$(L_{m_2}^{r_1}L_{m_3}^{r_2+r_3}\cdots L_{m_s}^{r_s}) \le C_3\deg_{R_1}(f^{\Delta m_2})\cdot\prod_{i=3}^{s}\deg_{R_i}(f^{\Delta m_i}).$$

Combining the above inequalities, we get

$$(L_{m_1}^{r_1}\cdots L_{m_s}^{r_s}) \ge \frac{E_1(\mathbf{m})E_2(\mathbf{m})}{C_3 C_{R_2}^{r_1}}\cdot\frac{\deg_{R_1}(f^{\Delta m_1+\Delta m_2})}{\deg_{R_1}(f^{\Delta m_1})\cdot\deg_{R_1}(f^{\Delta m_2})}\cdot\prod_{i=1}^{s}\deg_{R_i}(f^{\Delta m_i})$$

Set

$$E(\mathbf{m}) := \frac{E_1(\mathbf{m})E_2(\mathbf{m})}{C_3 C_{R_2}^{r_1}}\cdot\frac{\deg_{R_1}(f^{\Delta m_1+\Delta m_2})}{\deg_{R_1}(f^{\Delta m_1})\cdot\deg_{R_1}(f^{\Delta m_2})}.$$

Easily to check that $E(\mathbf{m})$ satisfies the desired property. ☐

Lemma 3.6. We have $\Sigma(\mu_i\mu_{i+1})_m \asymp_e \mu_i^m$ for every $i = 0, 1, \dots, d$.

Proof. We have

$$\begin{aligned} M(\mu_i\mu_{i+1})_m^d &= \sum_{j=0}^{d}\binom{d}{j}L_{2m}^j\cdot(\mu_i^m\mu_{i+1}^m L)^{d-j} \\ &\asymp_e \sum_{j=0}^{d}\mu_1^{2m}\cdots\mu_j^{2m}\cdot\mu_i^{m(d-j)}\mu_{i+1}^{m(d-j)} \\ &\asymp_e \mu_1^{2m}\cdots\mu_i^{2m}\cdot\mu_i^{m(d-i)}\mu_{i+1}^{m(d-i)} \end{aligned}$$

and

$$M(\mu_i\mu_{i+1})_m^{d-1}\cdot L_m = \sum_{j=0}^{d-1}\binom{d-1}{j}\left(L_{2m}^j\cdot L_m\cdot L^{d-1-j}\right)\cdot(\mu_i^m\mu_{i+1}^m)^{d-j-1}$$

$$
\begin{aligned}
&\asymp_e \sum_{j=0}^{d-1} \mu_1^{2m}\cdots\mu_j^{2m}\cdot\mu_{j+1}^{m}\cdot\mu_i^{m(d-1-j)}\mu_{i+1}^{m(d-j-1)} \\
&\asymp_e \mu_1^{2m}\cdots\mu_i^{2m}\cdot\mu_{i+1}^{m}\cdot\mu_i^{m(d-1-i)}\mu_{i+1}^{m(d-1-i)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Sigma(\mu_i\mu_{i+1})_m &= \frac{(L_{2m}+\mu_i^m\mu_{i+1}^m L)^d}{(L_{2m}+\mu_i^m\mu_{i+1}^m L)^{d-1}\cdot L_m} \\
&\asymp_e \frac{\mu_1^{2m}\cdots\mu_i^{2m}\cdot\mu_i^{m(d-i)}\mu_{i+1}^{m(d-i)}}{\mu_1^{2m}\cdots\mu_i^{2m}\cdot\mu_{i+1}^{m}\cdot\mu_i^{m(d-1-i)}\mu_{i+1}^{m(d-1-i)}} \\
&\asymp_e \mu_i^m.
\end{aligned}
$$

We are done. □

Theorem 3.7. For every $i = 0, 1, \dots, d$ and $\varepsilon \in (0,1)$, there exists $m_\varepsilon > 0$ such that for all $m \geq m_\varepsilon$,

$$L_{2m} - \varepsilon^m\mu_i^m L_m + \mu_i^m\mu_{i+1}^m L$$

is big.

Proof. For every $\varepsilon \in (0,1)$, there exists $m_\varepsilon > 0$ such that for every $m \geq m_\varepsilon$, we have $\Sigma(\mu_i\mu_{i+1})_m > \varepsilon^m\mu_i^m$. By Siu's criterion, this implies that $M(\mu_i\mu_{i+1})_m - \varepsilon^m\mu_i^m L_m$ is big. □

3.3 Estimation of dynamical degrees

In this subsection, we will try to estimate λ_i using only finitely many mixed degrees.

Given numbers $\alpha_1 \geq \dots \geq \alpha_d \in \mathbb{R}_{>0}$, and $\delta, \varepsilon \in (0,1)$. Set $\beta_i = \alpha_1\cdots\alpha_i$. We use $\mathfrak{A} = (\alpha_1, \dots, \alpha_d; \delta, \varepsilon)$ to denote the above data. Set

$$\Theta(\mathfrak{A}; i)_m := C_{\mathrm{mix}}\cdot\delta^m\cdot\sum_{j=0}^{i-1}\frac{\deg_{i-j-1}(f^m)}{\varepsilon^{(i-j)m}\beta_{i-j-1}^m}\cdot\alpha_i^m\deg_{i-1}(f^m),$$

where C_{mix} is the constant in Proposition 3.5. We will see this number appears naturally in the estimation of λ_i (eq. (2)). When choosing $\mathfrak{A} = (\mu_1, \dots, \mu_d; \delta, \varepsilon)$, we have

$$\Theta(\mathfrak{A}; i)_m \asymp_e \left(\varepsilon^{-i}\delta\lambda_i\right)^m.$$

Definition 3.8. We define the following conditions for $i = 1, \dots, d$:

- condition $I(\mathfrak{A}, i, m)$: $\Sigma(\alpha_i\alpha_{i-j}\delta; L_m^j)_m > \varepsilon^m\alpha_{i-j}^m$ for every $j = 0, 1, \dots, i-1$;
- condition $J(\mathfrak{A}, i, m)$: $\beta_i^m\varepsilon^{mi} \geq \varepsilon^{-mi}\Theta(\mathfrak{A}; i)_m + \beta_i^m\varepsilon^{2mi}$;
- condition $K(\mathfrak{A}, i, m)$: $\deg_i(f^{2m}) - \varepsilon^{-mi}\Theta(\mathfrak{A}; i)_m\cdot\deg_i(f^m) > 0$.

Note that all of the above conditions depend only on the intersection numbers of L_{2m}, L_m, L.

Theorem 3.9. Given a data $\mathfrak{A} = (\alpha_1, \dots, \alpha_d; \delta, \varepsilon)$. If conditions $I(\mathfrak{A}, i, m)$, $J(\mathfrak{A}, i, m)$ and $K(\mathfrak{A}, i, m)$ hold, then $\lambda_i \geq \varepsilon^{2i}\beta_i$.

Proof.

Step 1. Using condition $I(\mathfrak{A}, i, m)$ to get eq. (1) for all $n \geq 0$.

We may suppose that X is normal and L_m is ample on X. Fix $n \geq 0$. There exists another normal birational model X' of X such that f^{nm} is a morphism $X' \to X$. By generic flatness, there exists a non-empty open subset U of X such that f^{nm} is flat over U. By Bertini's theorem, there is a normal subvariety Z of X given by cutting by j general members of $|L_m|$ intersecting U. Denote by $W_1, \dots, W_r$ the irreducible components of $(f^{nm})^{-1}(Z)$. Since f^{nm} is flat at the generic point of Z, we have $(f^{nm})^*[Z] = \sum_i a_i[W_i]$ as cycles for some positive integers a_i's.

By Siu's criterion and condition $I(\mathfrak{A}, i, m)$, the divisor $(L_{2m} + \alpha_i^m \alpha_{i-j}^m \delta^m L - \varepsilon^m \alpha_{i-j}^m L_m)|_Z$ is big on Z for every $j = 0, 1, \dots, i-1$. Then

$$(f_t^{nm})^*\big((L_{2m} + \alpha_i^m \alpha_{i-j}^m \delta^m L - \varepsilon^m \alpha_{i-j}^m L_m)|_Z\big)$$

is big on W_t for every $j = 0, 1, \dots, i-1$ and $t = 1, \dots, r$. Suppose that it is represented by an effective $\mathbb{R}$-divisor $D_{t,j}$ on W_t. Hence

$$(f^{nm})^*\big((L_{2m} + \alpha_i^m \alpha_{i-j}^m \delta^m L - \varepsilon^m \alpha_{i-j}^m L_m) \cdot L_m^j\big) = \sum_t a_t (\iota_t)_* D_{t,j} \in \mathrm{CH}^{j+1}(X')_{\mathbb{R}}$$

is represented by an effective cycle, where $\iota_t : W_t \to X'$ is the natural inclusion. Intersecting with $L_{(n+2)m}^{i-j-1} \cdot L^{d-i}$, we get

$$L_{(n+2)m}^{i-j} \cdot L_{(n+1)m}^{j} \cdot L^{d-1} + \alpha_i^m \alpha_{i-j}^m \delta^m L_{(n+2)m}^{i-j-1} \cdot L_{(n+1)m}^{j} \cdot L_{nm} \cdot L^{d-i} \geq \varepsilon^m \alpha_{i-j}^m L_{(n+2)m}^{i-j-1} \cdot L_{(n+1)m}^{j+1} \cdot L^{d-i}. \quad (1)$$

Step 2. Show eq. (3) of $\deg_i f^{nm}$.

Divide eq. (1) by $\varepsilon^{(i-j)m}\beta_{i-j}^m$, we get

$$\frac{L_{(n+2)m}^{i-j} \cdot L_{(n+1)m}^{j} \cdot L^{d-1}}{\varepsilon^{(i-j)m}\beta_{i-j}^m} + \frac{\alpha_i^m \delta^m L_{(n+2)m}^{i-j-1} \cdot L_{(n+1)m}^{j} \cdot L_{nm} \cdot L^{d-i}}{\varepsilon^{(i-j)m}\beta_{i-j-1}^m} \geq \frac{L_{(n+2)m}^{i-j-1} \cdot L_{(n+1)m}^{j+1} \cdot L^{d-i}}{\varepsilon^{(i-j-1)m}\beta_{i-j-1}^m}.$$

By Proposition 3.5, we have

$$L_{(n+2)m}^{i-j-1} \cdot L_{(n+1)m}^{j} \cdot L_{nm} \cdot L^{d-i} \leq C_{\mathrm{mix}} \cdot \deg_{i-j-1}(f^m) \cdot \deg_{i-1}(f^m) \cdot \deg_i(f^{nm}).$$

Then the above inequality can be rewritten as

$$\begin{aligned} &\frac{L_{(n+2)m}^{i-j} \cdot L_{(n+1)m}^{j} \cdot L^{d-1}}{\varepsilon^{(i-j)m}\beta_{i-j}^m} - \frac{L_{(n+2)m}^{i-j-1} \cdot L_{(n+1)m}^{j+1} \cdot L^{d-i}}{\varepsilon^{(i-j-1)m}\beta_{i-j-1}^m} \\ &\qquad \geq -\left(C_{\mathrm{mix}} \cdot \delta^m \cdot \frac{\deg_{i-j-1}(f^m)}{\varepsilon^{(i-j)m}\beta_{i-j-1}^m} \cdot \alpha_i^m \deg_{i-1}(f^m)\right) \cdot \deg_i(f^{nm}). \end{aligned} \quad (2)$$

Summing over $j = 0, 1, \dots, i-1$, we get

$$\frac{L_{(n+2)m}^{i} \cdot L^{d-i}}{\varepsilon^{im}\beta_i^m} - (L_{(n+1)m}^{i} \cdot L^{d-i}) \geq -\Theta(\mathfrak{A}; i)_m \cdot \deg_i(f^{nm}).$$

That is,

$$\deg_i f^{(n+2)m} - \varepsilon^{im}\beta_i^m \deg_i f^{(n+1)m} + \varepsilon^{im}\beta_i^m \Theta(\mathfrak{A}; i)_m \cdot \deg_i(f^{nm}) \geq 0. \quad (3)$$

Step 3. Finish the proof by Lemma 3.4.

The condition $J(\mathfrak{A}, i, m)$ implies that

$$\deg_i f^{(n+2)m} + \left(\varepsilon^{2im}\beta_i^m \cdot \varepsilon^{-mi}\Theta(\mathfrak{A}; i)_m\right) \cdot \deg_i(f^{nm}) \geq \left(\varepsilon^{2im}\beta_i^m + \varepsilon^{-mi}\Theta(\mathfrak{A}; i)_m\right) \cdot \deg_i f^{(n+1)m}.$$

While the condition $K(\mathfrak{A}, i, m)$ gives the initial condition $\deg_i f^{2m} > \varepsilon^{-mi}\Theta(\mathfrak{A}; i)_m \cdot \deg_i(f^m)$. Then by Lemma 3.4, we have $\deg_i f^{nm} \succcurlyeq_e (\varepsilon^{2im}\beta_i^m)^n$. Hence $\lambda_i \geq \varepsilon^{2i}\beta_i$. ☐

Theorem 3.10. Suppose that $\mu_i > \mu_{i+1}$. For every $\varepsilon_0 \in (0,1)$, there exists $\delta \in (0,1)$ and $\varepsilon \in (\varepsilon_0, 1)$ such that for every sufficiently large m, conditions $I(\mathfrak{A}, i, m)$, $J(\mathfrak{A}, i, m)$ and $K(\mathfrak{A}, i, m)$ hold for $\mathfrak{A} = (\mu_1, \dots, \mu_d; \delta, \varepsilon)$.

Proof. Choose $\delta \in (0,1)$ such that $\delta\mu_i > \mu_{i+1}$ and $\varepsilon \in (\varepsilon_0, 1) \cap (\delta^{1/3d}, 1)$.

By Lemma 3.11, there exists $m_1 > 0$ such that for every $m \geq m_1$, we have $\Sigma(\mu_i\mu_{i-j}\delta; L_m^j)_m > \varepsilon^m\mu_{i-j}^m$ for every $j = 0, 1, \dots, i-1$. Hence condition $I(\mathfrak{A}, i, m)$ holds for every $m \geq m_1$.

For condition $J(\mathfrak{A}, i, m)$, we have

$$\begin{aligned}
\varepsilon^{-mi}\Theta(\mathfrak{A}; i)_m + \lambda_i^m\varepsilon^{2mi} &\asymp_e \varepsilon^{-2mi} \cdot \delta^m \cdot \lambda_i^m + \varepsilon^{2mi}\lambda_i^m \\
&\prec_e \varepsilon^{3dm-2im} \cdot \lambda_i^m + \varepsilon^{mi}\lambda_i^m \\
&\preccurlyeq_e \varepsilon^{2mi}\varepsilon^{mi}\lambda_i^m.
\end{aligned}$$

Then there exists $m_2 > 0$ such that for every $m \geq m_2$, the condition $J(\mathfrak{A}, i, m)$ holds.

For condition $K(\mathfrak{A}, i, m)$, we have

$$\deg_i(f^{2m}) \asymp_e \lambda_i^{2m} \succ_e \varepsilon^m \cdot \lambda_i^{2m} \succcurlyeq_e \varepsilon^{3dm-2im} \cdot \lambda_i^m \cdot \lambda_i^m \succcurlyeq_e \varepsilon^{-mi}\Theta(\mathfrak{A}; i)_m \cdot \deg_i(f^m).$$

Hence there exists $m_3 > 0$ such that for every $m \geq m_3$, the condition $K(\mathfrak{A}, i, m)$ holds.

Then for every $m \geq \max\{m_1, m_2, m_3\}$, conditions $I(\mathfrak{A}, i, m)$, $J(\mathfrak{A}, i, m)$ and $K(\mathfrak{A}, i, m)$ hold. ☐

Lemma 3.11. Suppose that $\mu_i > \mu_{i+1}$. The for every $\delta \in (0,1)$ with $\delta\mu_i > \mu_{i+1}$, we have

$$\Sigma(\mu_i\mu_{i-j}\delta; L_m^j)_m \asymp_e \mu_i^m \quad \text{for every } j = 0, 1, \dots, i-1.$$

Proof. Similar to the proof of Lemma 3.6. We have

$$\begin{aligned}
&M(\mu_i\mu_{i-j}\delta)_m^{d-j} \cdot L_m^j \\
&= \sum_{k=0}^{d-j} \binom{d-j}{k} L_{2m}^k \cdot (\mu_{i-j}^m\mu_i^m\delta^m L)^{d-j-k} \cdot L_m^j \\
&\asymp_e \sum_{k=0}^{d-j} \mu_1^{2m} \cdots \mu_k^{2m} \cdot \mu_{k+1}^m \cdots \mu_{k+j}^m \cdot \mu_{i-j}^{m(d-j-k)}\mu_i^{m(d-j-k)}\delta^{m(d-j-k)}.
\end{aligned}$$

Set

$$a_k := \mu_1^2 \cdots \mu_k^2 \cdot \mu_{k+1} \cdots \mu_{k+j} \cdot \mu_i^{d-j-k}\mu_{i-j}^{d-j-k}\delta^{d-j-k}.$$

Then, by the choice of δ, we have

$$\frac{a_{k+1}}{a_k} = \frac{\mu_{k+1} \cdot \mu_{k+1+j}}{\mu_{i-j} \cdot \mu_i} \cdot \delta^{-1} \begin{cases} > 1, & k \leq i-j-1, \\ < 1, & k \geq i-j. \end{cases} \tag{4}$$

Hence the maximum of a_k's is attained at $k = i - j$. Then

$$M(\mu_i\mu_{i-j}\delta)_m^{d-j} \cdot L_m^j \asymp_e \mu_1^{2m} \cdots \mu_{i-j}^{2m} \cdot \mu_{i-j+1}^m \cdots \mu_i^m \cdot \mu_i^{m(d-i)} \mu_{i-j}^{m(d-i)} \delta^{m(d-i)}. \tag{5}$$

Similarly, we have

$$M(\mu_i\mu_{i-j}\delta)_m^{d-j-1} \cdot L_m^{j+1} \asymp_e \sum_{k=0}^{d-j-1} \mu_1^{2m} \cdots \mu_k^{2m} \cdot \mu_{k+1}^m \cdots \mu_{k+j+1}^m \cdot \mu_{i-j}^{m(d-j-1-k)} \mu_i^{m(d-j-1-k)} \delta^{m(d-j-1-k)}.$$

Set

$$b_k := \mu_1^2 \cdots \mu_k^2 \cdot \mu_{k+1} \cdots \mu_{k+j+1} \cdot \mu_i^{d-j-1-k} \mu_{i-j}^{d-j-1-k} \delta^{d-j-1-k}.$$

Then we have

$$\frac{b_{k+1}}{b_k} = \frac{\mu_{k+1} \cdot \mu_{k+1+j+1}}{\mu_{i-j} \cdot \mu_i} \cdot \delta^{-1} \begin{cases} \geq \dfrac{\mu_{k+1+1} \cdot \mu_{k+1+j+1}}{\mu_{i-j} \cdot \mu_i} \cdot \delta^{-1} = \dfrac{a_{k+2}}{a_{k+1}} > 1, & k < i-j-1; \\ = \dfrac{\mu_{i-j} \cdot \mu_{i+1}}{\mu_{i-j} \cdot \mu_i} \cdot \delta^{-1} < 1, & k = i-j-1; \\ \leq \dfrac{\mu_{k+1} \cdot \mu_{k+1+j}}{\mu_{i-j} \cdot \mu_i} \cdot \delta^{-1} = \dfrac{a_{k+1}}{a_k} < 1, & k \geq i-j. \end{cases} \tag{6}$$

Hence the maximum of b_k's is attained at $k = i - j - 1$. Then

$$M(\mu_i\mu_{i-j}\delta)_m^{d-j-1} \cdot L_m^{j+1} \asymp_e \mu_1^{2m} \cdots \mu_{i-j-1}^{2m} \cdot \mu_{i-j}^m \cdots \mu_i^m \cdot \mu_i^{m(d-i)} \mu_{i-j}^{m(d-i)} \delta^{m(d-i)}. \tag{7}$$

Combining eqs. (5) and (7), we get

$$\begin{aligned} \Sigma(\mu_i\mu_{i-j}\delta; L_m^j)_m &= \frac{M(\mu_i\mu_{i-j}\delta)_m^{d-j} \cdot L_m^j}{M(\mu_i\mu_{i-j}\delta)_m^{d-j-1} \cdot L_m^{j+1}} \\ &\asymp_e \frac{\mu_1^{2m} \cdots \mu_{i-j-1}^{2m}\mu_{i-j}^{2m} \cdot \mu_{i-j+1}^m \cdots \mu_i^m \cdot \mu_i^{m(d-i)} \mu_{i-j}^{m(d-i)} \delta^{m(d-i)}}{\mu_1^{2m} \cdots \mu_{i-j-1}^{2m} \cdot \mu_{i-j}^m \mu_{i-j+1}^m \cdots \mu_i^m \cdot \mu_i^{m(d-i)} \mu_{i-j}^{m(d-i)} \delta^{m(d-i)}} \\ &= \mu_{i-j}^m. \end{aligned}$$

We are done. ☐

Remark 3.12. The eqs. (4) and (6) are the places where we use the condition $\delta\mu_i > \mu_{i+1}$.

3.4 Lower bounds for λ_1 on surfaces

Yang: To be written.

4 Semi-continuity

In this section, we fix an integral noetherian base scheme S, a flat projective morphism $X \to S$ with integral fibers, and $f : X \dashrightarrow X$ a family of dominant rational maps over S. For each $s \in S$, we denote by X_s the fiber of X over s, and by $f_s : X_s \dashrightarrow X_s$ the induced rational map on the fiber.

4.1 Semi-continuity of mixed degrees

Proposition 4.1. Fix a π-ample divisor L on X and integers $m_1, \dots, m_d \in \mathbb{Z}_{\geq 0}$. Then the function

$$s \mapsto (f_s^{m_1})^* L_s \cdot \cdots \cdot (f_s^{m_d})^* L_s$$

is lower semi-continuous on S.

Proof. Let $\Gamma_{f,\mathbf{m}} \subset X \times_S \cdots \times_S X$ be the closure of the image of the rational map $(f^{m_1}, \dots, f^{m_d}) : X \dashrightarrow X \times_S \cdots \times_S X$. By Gruson-Raynaud's flattening theorem (Theorem 2.10), there exists a blowup $S' \to S$ such that the strict transform $\Gamma'_{f,\mathbf{m}}$ of $\Gamma_{f,\mathbf{m}}$ is flat over S'. Let X', L', f' be the base change of X, L, f along $S' \to S$ respectively. For every $s' \in S'$ with image $s \in S$, the date $(X'_{s'}, L'_{s'}, f'_{s'})$ is the base change of (X_s, L_s, f_s). Hence the semi-continuity on S follows from the semi-continuity on S'. By replacing S with S', we may assume that $\Gamma_{f,\mathbf{m}}$ is flat over S.

For every $s \in S$, let $\Gamma_{f_s,\mathbf{m}}$ be the closure of the image of the rational map $(f_s^{m_1}, \dots, f_s^{m_d}) : X_s \dashrightarrow X_s \times \cdots \times X_s$. Denote by $\pi_i : X \times_S \cdots \times_S X \to X$ the projection to the i-th factor. Then we have

$$(f_s^{m_1})^* L_s \cdot \cdots \cdot (f_s^{m_d})^* L_s = \pi_1^* L \cdot \cdots \cdot \pi_d^* L \cdot [\Gamma_{f_s,\mathbf{m}}].$$

Over the generic point η of S, we have $(\Gamma_{f,\mathbf{m}})_\eta = \Gamma_{f_\eta,\mathbf{m}}$. Over a special point $s \in S$, $(\Gamma_{f,\mathbf{m}})_s$ may have several irreducible components, and $\Gamma_{f_s,\mathbf{m}}$ is one of the irreducible components of $(\Gamma_{f,\mathbf{m}})_s$. Since $\Gamma_{f,\mathbf{m}}$ is flat over S and intersection numbers are constant in flat families, we have

$$\begin{aligned}
(f_s^{m_1})^* L_s \cdot \cdots \cdot (f_s^{m_d})^* L_s &= \pi_1^* L \cdot \cdots \cdot \pi_d^* L \cdot [\Gamma_{f_s,\mathbf{m}}] \\
&\leq \pi_1^* L \cdot \cdots \cdot \pi_d^* L \cdot [(\Gamma_{f,\mathbf{m}})_s] \\
&= \pi_1^* L \cdot \cdots \cdot \pi_d^* L \cdot [(\Gamma_{f,\mathbf{m}})_\eta] \\
&= \pi_1^* L \cdot \cdots \cdot \pi_d^* L \cdot [\Gamma_{f_\eta,\mathbf{m}}] \\
&= (f_\eta^{m_1})^* L_\eta \cdot \cdots \cdot (f_\eta^{m_d})^* L_\eta.
\end{aligned}$$

And since the generic fiber is (geometrically) irreducible, there exists a non-empty open subset $U \subseteq S$ such that for every $s \in U$, we have $(\Gamma_{f,\mathbf{m}})_s$ is irreducible and hence equal to $\Gamma_{f_s,\mathbf{m}}$. Then for every $s \in U$, the above inequalities are equalities. By Lemma 2.20, we are done. ☐

Example 4.2. Let $S = \operatorname{Spec} \mathbb{Z}$, $X = \mathbb{P}^2_{\mathbb{Z}}$, $L = \mathcal{O}(1)$, and $f : X \dashrightarrow X$ is defined by

$$[x : y : z] \mapsto [xz : yz + 2xy : z^2].$$

Then the function $s \mapsto \deg_1(f_s) = L_s \cdot (f_s^* L_s)$ is not constant on S. Indeed, we have $\deg_1(f_{(2)}) = 1$ and $\deg_1(f_{(p)}) = 2$ for every prime $p \neq 2$.

Let $\Gamma \subset X \times_S X = \mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}$ be the graph of f. On $\mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}$, we use the coordinates $[x : y : z]$ and $[u : v : w]$ for the first and second factor respectively. Set $V = V(u(yz + 2xy) - vxz, w(yz + 2xy) - vz^2, wx - uz)$. Then Γ is an irreducible component of V. Consider the first projection $\pi_1 : V \to X$. Over the open subset $z \neq 0$ of X, π_1 is an isomorphism. When $x, y \neq 0$ and $z = 0$, π_1 is also an isomorphism. Over the point $[1 : 0 : 0]$, the fiber of π_1 is a line in the second factor defined by $w = 0$. Over the point $[0 : 1 : 0]$, the fiber of π_1 is the full fiber $\{[0 : 1 : 0]\} \times \mathbb{P}^2$. Hence $V = \Gamma \cup (\{[0 : 1 : 0]\} \times \mathbb{P}^2)$. We need to find a function vanishing on Γ but not vanishing on

$\{[0:1:0]\} \times \mathbb{P}^2$. Note that $(2u+w)y - vz$ satisfies this condition. Hence Γ is given by

$$\Gamma = V(u(yz+2xy) - vxz, w(yz+2xy) - vz^2, wx - uz, (2u+w)y - vz).$$

Consider the first projection $\pi_1 : \Gamma \to X$. Over any point $s \in \operatorname{Spec} \mathbb{Z}$ different from (2), π_1 is an isomorphism over $\mathbb{P}^2_s \setminus \{[1:0:0],[0:1:0]\}$, and over this two points, the fiber of π_1 is a line. Hence π_1 is the blowup of X_s at these two points. Over the point (2), $\Gamma_{(2)}$ is given by

$$\begin{aligned}\Gamma_{(2)} &= V(uyz - vxz, wyz - vz^2, wx - uz, wy - vz)\\ &= V((uy - vx)z, wx - uz, wy - vz)\\ &= V(uy - vx, wx - uz, wy - vz) \cup V(z, w)\\ &= \Delta \cup (\{z = 0\} \times \{w = 0\}).\end{aligned}$$

It has two irreducible components, one is the diagonal Δ and the other is $\{z=0\} \times \{w=0\}$. Among them, Δ is the graph of $f_{(2)}$ and $\{z=0\} \times \{w=0\}$ is the "extra" component.

4.2 Semi-continuity of $\lambda_i(f)$

Theorem 4.3. Let S be an integral noetherian scheme, and $f : X \dashrightarrow X$ be a family of dominant rational maps over S. Then for every i, the function $s \mapsto \lambda_i(f_s)$ is lower semi-continuous on S.

Proof. Fix a π-ample divisor L on X. It determines a class $L_s \in \mathrm{N}^1(\mathfrak{X}_s)$ for every $s \in S$.

Let η be the generic point of S. For any $s \in S$, we have $\deg_i(f_s^m) \le \deg_i(f_\eta^m)$ for every m by the previous proposition (Proposition 4.1), which implies $\lambda_i(f_s) \le \lambda_i(f_\eta)$. Hence we only need to show that for every $\varepsilon_0 \in (0,1)$, there is an open subset $U \subseteq S$ such that for every $s \in U$, we have $\lambda_i(f_s) \ge \varepsilon_0 \lambda_i(f_\eta)$.

First assume that $\mu_i(f_\eta) > \mu_{i+1}(f_\eta)$. By Theorem 3.10, there exists $\varepsilon \in (\varepsilon_0^{1/2d}, 1)$ and $\delta \in (0,1)$, $m \ge 0$ such that the conditions $I(\mathfrak{A}_\eta, i, m)$, $J(\mathfrak{A}_\eta, i, m)$ and $K(\mathfrak{A}_\eta, i, m)$ hold, where $\mathfrak{A}_\eta = (\mu_1(f_\eta), \ldots, \mu_d(f_\eta); \delta, \varepsilon)$. Note that the conditions $I(\mathfrak{A}, i, m)$, $J(\mathfrak{A}, i, m)$ and $K(\mathfrak{A}, i, m)$ for (X_s, f_s, L_s) depend only on the intersection numbers of $(f_s^{2m})^* L_s, (f_s^m)^* L_s, L_s$. By Proposition 4.1, there exists a non-empty open subset $U \subseteq S$ such that for every $s \in U$, all intersection numbers of $(f_s^{2m})^* L_s, (f_s^m)^* L_s, L_s$ are constant on U. Hence for every $s \in U$, the conditions $I(\mathfrak{A}_\eta, i, m)$, $J(\mathfrak{A}_\eta, i, m)$ and $K(\mathfrak{A}_\eta, i, m)$ hold for (X_s, f_s, L_s). By Theorem 3.9, for every $s \in U$, we have

$$\lambda_i(f_s) > \varepsilon^{2i} \lambda_i(f_\eta) > \varepsilon_0^{i/d} \lambda_i(f_\eta) \ge \varepsilon_0 \lambda_i(f_\eta).$$

This finishes the proof in the case $\mu_i(f_\eta) > \mu_{i+1}(f_\eta)$.

Now we deal the general case. Choose a non-empty open subset $U \subseteq S$ such that for every $s \in U$, we have $\lambda_i(f_s) \ge \varepsilon_0 \lambda_i(f_\eta)$ for every i such that $\mu_i(f_\eta) > \mu_{i+1}(f_\eta)$. Let $a_s : [0,d] \to \mathbb{R}$ be the concave envelope of the points $(0,0), (1, \log \lambda_1(f_s)), \ldots, (d, \log \lambda_d(f_s))$. Note that a_η is a piecewise linear function with break points at exactly the integers i such that $\mu_i(f_\eta) > \mu_{i+1}(f_\eta)$. For two concave functions $a, b : [0,d] \to \mathbb{R}$ and a is piecewise linear with break points at exactly $x_1, \ldots, x_k$, we have $a \le b$ if and only if $a(x_i) \le b(x_i)$ for every i. Now applying this observation to a_s and $a_\eta + \log \varepsilon_0$ with $s \in U$, we conclude that $a_s \ge a_\eta + \log \varepsilon_0$ and hence $\lambda_i(f_s) \ge \varepsilon_0 \lambda_i(f_\eta)$ for every i. ☐

Example 4.4. Let $S = \operatorname{Spec}\mathbb{Z}$, $X = \mathbb{P}^2_{\mathbb{Z}}$, and $f : X \dashrightarrow X$ is defined by

$$[x : y : z] \mapsto [xy : xy - 2z^2 : yz + 3z^2].$$

Then we have $\lambda_1(f_{(0)}) = 2$ and $\lambda_1(f_{(p)}) < 2$ for every prime $p > 2$.

Yang: to be continued

5 Density of periodic points

5.1 General strategy

For a general rational self-map $f : X \dashrightarrow X$ defined over $\overline{\mathbb{F}_p}$, we will try to find intersection points of the graph of f and the graph of Frob_p^n for some $n > 0$, which will be periodic points of f. To find such points, we will use the Hrushovski-Lang-Weil estimate, which is a generalization of the classical Lang-Weil estimate for counting points on varieties over fields of positive characteristic.

Theorem 5.1 (Hrushovski-Lang-Weil estimate, ref. [Hru04, Theorem 1.1])**.** Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$ and X be an affine variety over $\Bbbk$. Let q be a power of p and $\mathrm{Frob}_{q,/\Bbbk} : X \to X^{(q)}$ be the relative Frobenius morphism. Let $S \subset X \times X^{(q)}$ be a subvariety of the same dimension as X such that the two projections $p_1 : S \to X$ and $p_2 : S \to X^{(q)}$ are both dominant and p_2 is quasi-finite. Then there is a constant c depending only on the degree of S and X such that for all $q > c$, we have

$$\#(S \cap \Gamma_{\mathrm{Frob}_{q,/\Bbbk}})(\Bbbk) = \frac{\deg(p_1)}{\deg(p_2)_{\mathrm{insep}}} q^{\dim X} + O(q^{\dim X - \frac{1}{2}}),$$

where $\Gamma_{\mathrm{Frob}_{q,/\Bbbk}}$ is the graph of $\mathrm{Frob}_{q,/\Bbbk}$ and $\deg(p_2)_{\mathrm{insep}}$ is the inseparable degree of p_2.

The statement of Theorem 5.1 is stated in full generality but a little bit technical, and we will only use the following special case.

Lemma 5.2 (ref. [Hru04, Corollary 1.2])**.** Let X be an affine variety defined over $\mathbb{F}_q$ and $S \subset X \times X$ be a subvariety over $\overline{\mathbb{F}_q}$ such that the two projections $S \to X$ are both dominant. Then for any proper closed subset $W \subset X$, for large enough m, there exists a point $x \in X(\overline{\mathbb{F}_q})$ such that $(x, \mathrm{Frob}_q^m(x)) \in S$ and $x \notin W$.

Lemma 5.3. Let X be a variety over $\overline{\mathbb{F}_p}$ and $f : X \dashrightarrow X$ be a dominant rational self-map. Then the set of periodic points of f is Zariski dense in X.

Proof. Let U be an non-empty affine open subset of X. We only need to show that there exists a periodic point of f in U. Let q be a power of p such that U and f is defined over $\mathbb{F}_q$. Set Γ_f to be the graph of f in $U \times U$ and Γ_{Frob_q} to be the graph of Frob_q in $U \times U$.

Choose $W \subset U$ to be a proper closed subset such that f is defined on $U \setminus W$ and $f(U \setminus W) \subset U$. By the Hrushovski-Lang-Weil estimate, for large enough m, there exists a point $x \in U(\overline{\mathbb{F}_q})$ such that $(x, \mathrm{Frob}_q^m(x)) \in \Gamma_f$ and $x \notin W$. This means that $f(x) = \mathrm{Frob}_q^m(x)$.

Since X is defined over $\mathbb{F}_q$, there exists $n > 0$ such that $\mathrm{Frob}_q^{mn}(x) = x$. Note that f is defined over $\mathbb{F}_q$, we have

$$f^n(x) = f^{n-1}(f(x)) = f^{n-1}(\mathrm{Frob}_q^m(x)) = \mathrm{Frob}_q^m(f^{n-1}(x)) = \cdots = \mathrm{Frob}_q^{mn}(x) = x.$$

We are done. ☐

Definition 5.4. We say a periodic point x of f is *isolated* if there exists a period r of x such that x is an isolated fixed point of f^r.

An important fact is that isolated periodic points can be lifted to generic fibers in a family, which allows us to reduce the density of periodic points to the density of isolated periodic points on special fibers.

Lemma 5.5. Let $S = \operatorname{Spec} R$ be spectrum of a discrete valuation ring with special point s and generic point η. Let X be a scheme flat and of finite type over S, and $f : X \dashrightarrow X$ be a rational self-map over S. Assume that the special fiber X_s is irreducible and the restriction $f_s : X_s \dashrightarrow X_s$ is a dominant rational self-map.
If $x \in X_s$ is an isolated periodic point of f_s and X_s is regular at x, then there exists a periodic point $y \in X_\eta$ of f_η such that the closure of y in X contains x.

Proof. Replace f by a suitable iterate, we may assume that x is an isolated fixed point of f_s. Let Γ_f be the graph of f in $X \times_S X$ and Δ_X be the diagonal of $X \times_S X$. Since X_s is regular at x, X is flat over S at x, and S is regular, we have X is regular at x. Then the embedding $\Delta_X \hookrightarrow X \times_S X$ is a regular embedding near x. By Lemma 2.9, we have

$$\dim_x(\Gamma_f \cap \Delta_X) \geq \dim_x \Gamma_f + \dim_x \Delta_X - \dim_x(X \times_S X) = 1.$$

Let Z be an irreducible component of $\Gamma_f \cap \Delta_X$ containing (x, x), then $\dim Z \geq 1$. Since x is an isolated fixed point of f_s, the special fiber of Z is just the point x. Then $Z \to S$ is dominant. In particular, the generic fiber of Z over S is non-empty. Let y be a point in the generic fiber of Z, then y is a periodic point of f_η and the closure of y in X contains x. ☐

Example 5.6. When x is a non-isolated periodic point of f_s, we may not be able to lift x to a periodic point of f_η in general. For example, let $S = \operatorname{Spec} \mathbb{Z}_p$ and $X = \mathbb{A}^1_S$. Consider the morphism $f : X \to X$ defined by $f(x) = x + p$. Over the special fiber, f_s is the identity map, and every point is a periodic point. However, over the generic fiber, f_η has no periodic point.

We have $X \times_S X = \operatorname{Spec} \mathbb{Z}_p[x, y]$, $\Delta_X = \{x = y\} = \operatorname{Spec} \mathbb{Z}_p[x, y]/(x - y)$ and $\Gamma_f = \{y = x + p\} = \operatorname{Spec} \mathbb{Z}_p[x, y]/(y - x - p)$. Then

$$\Delta_X \cap \Gamma_f = \operatorname{Spec} \mathbb{Z}_p[x, y]/(x - y, y - x - p) = \operatorname{Spec} \mathbb{F}_p[x].$$

Its special fiber is $\operatorname{Spec} \mathbb{F}_p[x]$, which is one-dimensional, and its generic fiber is empty.

5.2 For cohomologically hyperbolic maps

Given a dominant rational self-map $f : X \dashrightarrow X$ and $U \subset X$, we denoted by

$$\mathrm{Per}_U(f) := \{x \in U : f^r(x) = x \text{ for some } r > 0 \text{ and } f^n(x) \in U \text{ for all } n \geq 0\}$$

the set of periodic points of f whose orbits are contained in U.

Remark 5.7. The reason to introduce this notation is that in the proof of Proposition 5.8, we need eq. (8) to hold for all $n \geq 0$. If we only consider $\mathrm{Per}(f) \cap U$, then we can only get eq. (8) for n being a period of C, which is not enough to get a contradiction.

Proposition 5.8. Let X be a variety and $f : X \dashrightarrow X$ be a rational self-map. Suppose that f is cohomologically hyperbolic, then there exists an open dense subset U of X such that every periodic point in $\mathrm{Per}_U(f)$ is isolated.

Proof. Set $\mu_k = \mu_k(f)$ for $1 \leq k \leq d$ and $\mu_{d+1} = 0$ by convention. There exists i such that $\mu_i > 1 > \mu_{i+1}$ by the definition of cohomological hyperbolicity. Choose $a, b > 0$ such that $\mu_i > a > 1 > b > \mu_{i+1}$ and $a \cdot b = \mu_i \cdot \mu_{i+1}$. Then there exists $\varepsilon > 0$ such that $\varepsilon^m \mu_i^m > a^m + b^m$ for all $m > m_1$ for some $m_1 > 0$. By Theorem 3.7, the divisor

$$(f^{2m})^* L - \varepsilon^m \mu_i^m (f^m)^* L + \mu_i^m \mu_{i+1}^m L$$

is big for some ample line bundle L and all $m > m_2$ for some $m_2 > 0$. Then fix a large enough $m > \max\{m_1, m_2\}$, and let

$$M := (f^{2m})^* L - (a^m + b^m)(f^m)^* L + a^m b^m L.$$

Then M is big. In particular, M is numerically equivalent to an effective $\mathbb{R}$-divisor $\sum a_j D_j$ with $a_j > 0$ and D_j prime divisors. Set $Z = \bigcup D_j$ to be the support of M.

Let $U = X \setminus Z$ be the complement of Z in X, which is an open dense subset of X. We claim that every periodic point of f in U is isolated. Otherwise, there exists a curve C such that $f^n(C) \cap U \neq \varnothing$ for all $n \geq 0$ and $f^r|_C = \mathrm{id}_C$ for some $r > 0$. Since $f^n(C)$ intersects U for all $n \geq 0$, we have

$$(f^{nm})^* M \cdot C = M \cdot f_*^{nm} C \geq 0, \quad \forall n \geq 0. \tag{8}$$

Then we have

$$(f^{(n+2)m})^* L \cdot C - (a^m + b^m)(f^{(n+1)m})^* L \cdot C + a^m b^m (f^{nm})^* L \cdot C \geq 0.$$

Let $x_n = (f^{nm})^* L \cdot C$, then x_n satisfies the linear recurrence relation

$$x_{n+2} - (a^m + b^m) x_{n+1} + a^m b^m x_n \geq 0.$$

Since $b < 1$ and $x_n = (f^{nm})^* L \cdot C$ takes values in a finite set for $n \geq 0$, there exists n_0 such that $x_{n_0+1} > b^m x_{n_0}$. Hence $(f^{nm})^* L \cdot C = x_n > a^{nm}$ for $n \gg 0$ by Lemma 3.4. In particular, $x_n = L \cdot (f^{nm})_* C$ is unbounded as $n \to \infty$, which contradicts to the fact that $f^r|_C = \mathrm{id}_C$. ☐

Theorem 5.9. Let X be a variety and $f : X \dashrightarrow X$ be a dominant rational self-map. If f is cohomologically hyperbolic, then for every dense open subset U of X, $\mathrm{Per}_U(f)$ is Zariski dense in X. In particular, $\mathrm{Per}(f)$ is Zariski dense in X.

Proof. We may assume that $X = U$ is over a finitely generated field $\mathbf{k}$. Hence we only need to show that $\mathrm{Per}(f)$ is Zariski dense in X. The cost is that we can assume that X is projective.

When $\overline{\mathbf{k}} = \overline{\mathbb{F}_p}$, this is Lemma 5.3. For general case, by spreading out, we may assume that there is a Dedekind domain R with infinite many closed points and $\operatorname{Frac} R = \mathbf{k}$ and a family $(\pi : \mathcal{X} \to \operatorname{Spec} R, \mathscr{f} : \mathcal{X} \dashrightarrow \mathcal{X})$ of dominant rational self-maps such that its geometric generic fiber is (X, f). By Theorem B, after shrinking $\operatorname{Spec} R$ if necessary, we may assume that $(\mathcal{X}_s, \mathscr{f}_s)$ is cohomologically hyperbolic for every closed point $s \in \operatorname{Spec} R$. Fix a closed point $s \in \operatorname{Spec} R$ and replace R by the localization of R at s, we may assume that $\operatorname{Spec} R$ is the spectrum of a discrete valuation ring with special point s and generic point η.

By induction, we may assume that the theorem holds for $(\mathcal{X}_s, \mathscr{f}_s)$. Since $\mathscr{f}_s$ is cohomologically hyperbolic, there exists an open dense subset U of $\mathcal{X}_s$ such that every periodic point in $\operatorname{Per}_U(\mathscr{f}_s)$ is isolated and $\mathcal{X}_s$ is regular at that point by Proposition 5.8. Suppose the contrary that the set of periodic points of $\mathscr{f}_\eta$ lies in a proper closed subset Z_η of $\mathcal{X}_\eta$. Let $\mathcal{Z}$ be the closure of Z_η in $\mathcal{X}$, then $\mathcal{Z}$ is a proper closed subset of $\mathcal{X}$. Hence $\dim \mathcal{Z}_s < \dim \mathcal{Z} < \dim \mathcal{X}$ and then $\dim \mathcal{Z}_s \leq \dim \mathcal{X} - 2$. Hence $\mathcal{Z}_s$ is a proper closed subset of $\mathcal{X}_s$. Choose a periodic point $x_s \in \operatorname{Per}_U(\mathscr{f}_s)$ such that $x_s \notin \mathcal{Z}_s$ and lift x_s to a periodic point x_η of $\mathscr{f}_\eta$ in $\mathcal{X}_\eta$ by Lemma 5.5. Then x_η is a periodic point of $\mathscr{f}_\eta$ and the closure of x_η in $\mathcal{X}$ contains x_s, which is not contained in $\mathcal{Z}$. Hence x_η is not contained in Z_η, which is a contradiction. □

6 Application to the Kawaguchi-Silverman conjecture

Yang: To be written.

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